ANALYSIS EXAM January 2004

1. a) Let $\{f_n\}$ be a sequence of real-valued functions defined on a subset E of the real line. Give the definition of uniform convergence of the sequence $\{f_n\}$ to a function f.
   b) Let $\{f_n\}$ be a sequence of continuous real-valued functions defined on the interval $[0, 1]$. Prove that if $\{f_n\}$ converges uniformly to f on the interval $[0, 1]$, then f is continuous on $[0, 1]$.

2. a) For a sequence $\{a_n\}$ of real numbers, give a definition of $\limsup a_n$.
   b) Let $\mathcal{A}$ be a σ-algebra of subsets of the real line. Let $\{f_n\}$ be a sequence of real-valued functions on the real line. Suppose that for each n the set $E_n = \{x \in \mathbf{R} : f_n(x) > 0\}$ belongs to the σ-algebra $\mathcal{A}$. Denote $f(x) = \limsup f_n(x)$. Show that the set $E = \{x \in \mathbf{R} : f(x) > 0\}$ belongs to the σ-algebra $\mathcal{A}$.

3. Let E be a measurable set, and $mE < \infty$. Prove that, for every $\varepsilon > 0$, there is an open set O such that $E \subset O$ and $m(O \setminus E) < \varepsilon$.
   Here m denotes the Lebesgue measure on $\mathbf{R}$.

4. Let f and g be measurable real-valued functions on a measurable set E. Define a function h on E by setting $h(x) = \max\{f(x), g(x)\}$ for all $x \in E$. Prove that h is measurable.

5. a) Give the definitions of the spaces $L^p[0, 1]$, $1 \le p \le \infty$.
   b) Give an example of a sequence of integrable functions on $[0, 1]$ which is convergent in L^1-norm, but is not convergent almost everywhere (with respect to Lebesgue measure).

6. a) State monotone convergence theorem.
   b) Is monotonicity necessary? Justify your answer (with an example).

7. a) Let C be the Cantor ternary set.
   a) Show that for every open set $O \subset \mathbf{R}$, either $O \cap C$ is empty or $O \cap C$ is uncountable.
   b) Show that $m(C) = 0$, where m is the Lebesgue measure.

8. a) Define the functions of bounded variation on $[0, 1]$.
   b) If $f : [0, 1] \to \mathbf{R}$ is a function of bounded variation, is it true that $f(x) = \int_0^x f'(t)dt$ (here f' is the a.e. derivative of f)? Why? If your answer is negative, for what type of functions the answer is affirmative? (Name only.)

9. a) State the definitions of sets of first and second category in a metric space (M, d).
   b) Give an example of a set of first category which is uncountable.
   c) Are there sets of first category in $[0, 1]$ that have measure 1? (Yes or No.) No justification is necessary.

10. a) Show that if $f : [0, 1] \to \mathbf{R}$ is continuous, then it is uniformly continuous.
    b) Prove or disprove (by a counterexample): if $A \subset [0, 1]$ is closed, then $f(A)$ is also closed.