ANALYSIS EXAM May 2004

1. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers with $a_n > 0,\ b_n > 0$ for all $n \geq 1$. Show that $\limsup_n (a_n b_n) \leq (\limsup_n a_n)(\limsup_n b_n)$, provided that the product on the right hand side is not of the form $0 \times \infty$.

2. a) Give a definition of a measurable function.
   b) Let f be a real valued function on the real line. Which of the following statements are true? Justify your answers using the definition that you gave in part a).
   (i) if f is measurable, then f^2 is measurable.
   (ii) if f^2 is measurable, then f is measurable.

3. Let $f_1, f_2, \ldots$ be a sequence of nonnegative integrable functions on $[0,\, 1]$. Suppose that $\lim_{n\to\infty} \int_0^1 f_n = 0$. Denote $A_n = \{x \in [0,\, 1] : f_n(x) \geq 1\}$, and let $a_n = mA_n$ (here m, as usual denotes the Lebesgue measure). Show that $\lim_{n\to\infty} a_n = 0$.

4. a) State the Monotone Convergence Theorem. Is the monotonicity necessary, Why?
   b) Let $\{f_n\}$ be a sequence of integrable functions on $[0, 1]$ such that $0 \leq f_{n+1} \leq f_n$ a.e. for all n. Show that $f_n \to 0$ iff $\int f_n \to 0$.

5. a) Give a definition of a function of bounded variation on the interval $[a,\, b]$.
   b) Show that if f and g are functions of bounded variation on $[a,\, b]$, then their product $f \cdot g$ is also of bounded variation.

6. a) Give the definition of an absolutely continuous function.
   b) A function f on an interval $[a,\, b]$ is said to *satisfy the Lipschitz condition* if there is $M > 0$ such that $|f(s) - f(t)| \leq M|s - t|$ for all $s, t \in [a,\, b]$. Show that an absolutely continuous function satisfies the Lipschitz condition if and only if $|f'(x)|$ is bounded.

7. a) Prove that if $1 \leq p < q$, then $L_q[0, 1] \subset L_p[0, 1]$.
   b) Show by an example that the inclusion $L_2[0, 1] \subset L_1[0, 1]$ is proper.

8. a) Give the definition of a separable metric space.
   b) Prove that the space $L^\infty[0,\, 1]$ is **not** separable.

9. a) Give the definition of a set of first category in a metric space.
   b) Construct an example of a set of first category on the interval $[0,\, 1]$ (with usual metric) whose Lebesgue measure is 1.

10. a) Give a definition of a real valued uniformly continuous function on a metric space (X, d).
    b) Show that if a function $f : [0, 1] \to \mathbf{R}$ is continuous, then it is uniformly continuous.