Problems for Preliminary Exam
Applied Mathematics
January 2018

Part I
All problems have 10 points.

Problem 1. Consider the nonlinear autonomous system

$$\dot{x} = -y + x(x^2 + y^2 - 9), \quad \dot{y} = x + y(x^2 + y^2 - 9).$$

Transform the equations to polar coordinate and describe the nature of solution near $(0, 0)$. Are there any limit cycles?

Problem 2. Consider a first-order equation in $\mathbb{R}$

$$\dot{x} = f(t, x),$$

with $f(t, x)$ defined on $\mathbb{R} \times \mathbb{R}$. Suppose $xf(t, x) < 0$, for $|x| > R$, for a fixed $R \in \mathbb{R}$. Show that all solutions exist for all $t > 0$.

Problem 3. Assume that $\epsilon > 0$. Approximate the solutions of $\ddot{x} + x + \epsilon x^3 = 0$, $x(0) = 1$, $\dot{x}(0) = 0$, up to order one.

Problem 4. (a) Pendulum equation without friction can be written as

$$\dot{x} = y, \quad \dot{y} = -a \sin x, \quad a > 0.$$

Suppose $E(x, y) = a(1 - \cos x) + \frac{1}{2}y^2$. Find a domain for which $E(x, y)$ is positive definite and then use E as a Lyapunov function to analyze stability of solution near the origin.

(b) Consider Pendulum equation with friction

$$\dot{x} = y, \quad \dot{y} = -a \sin x - by, \quad a, b > 0.$$

Can you take the same Lyapunov function as in part (a)? If not, suggest an appropriate Lyapunov function and analyze the stability of solution near the origin (do the same if you think Lyapunov function from part (a) is appropriate in this case).

Please turn over...

Problem 5. Suppose $L \in C^1(M, \mathbb{R})$. Show that the level set $L(x) = c$ is invariant under the flow if and only if the Lie derivative of L along the vector field vanishes on this level set.

Part II
All problems have 10 points.

Problem 6. Does there exist a positive function u harmonic in the ball $B_3(0,1) = \{x \in \mathbb{R}^3 \; : \; \|x\| \leq 1\}$ such that $u(0,0,0) = 3$ and $u(0,0,\frac{1}{2}) = 25$?

Problem 7. Let $u(x_1, x_2, t)$ be a solution in $\mathbb{R}^2 \times \mathbb{R}_+$ of the following Cauchy problem:

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2}, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = ((1 - x_1^2 - x_2^2)_+)^5,$$

where $(z)_+ = max\{0, z\}$ for all z. Find $\lim_{t\to\infty} tu(x_1, x_2, t)$.

Problem 8. Let u be a solution of the following Cauchy problem:

$$u_t = u_{xx}, \quad u(x,0) = e^{-x^2}.$$

Find $\lim_{t\to\infty} \int_0^\infty u(x,t)dx$.

Problem 9. Let u be a solution of the following Cauchy problem in $\mathbb{R} \times \mathbb{R}_+$:

$$u_t = u_{xx} - u, \qquad u(x,0) = \sin^2(x).$$

Find $\lim_{t\to\infty} e^t u(x,t)$.

Problem 10. Solve the following Cauchy problem in $\mathbb{R} \times \mathbb{R}_+$:

$$\frac{\partial u}{\partial x_1} + (2x_1 + u^3)\frac{\partial u}{\partial x_2} + u = 0, \quad u(0, x_2) = (x_2)^{1/3}.$$