Analysis Premilinary Exam
September 2009

1. Let X be a non-empty set, let $\mathcal{M} \subseteq \mathcal{P}(X)$ be an algebra and let $\mu : \mathcal{M} \to [0, \infty]$ be a finitely additive measure with the property that, for every decreasing sequence $\{E_n\} \subseteq \mathcal{M}$ with empty intersection ($\cap_{n=1}^{\infty} E_n = \emptyset$), we have that $\lim_{n\to\infty} \mu(E_n) = 0$. Prove that μ is countably additive.

2. (a) State the monotone convergence theorem.

   (b) Does the conclusion of the monotone convergence theorem hold for non-increasing sequences? If yes, justify it. If not, provide a counterexample.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz if there exists a constant $C > 0$ such that, for every $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq C|x - y|$.

   (a) Prove that every Lipschitz function is absolutely continuous.

   (b) Give an example of an absolutely continuous function that is not Lipschitz.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions. Prove that if f_n converges to 0 almost everywhere, then f_n converges to 0 in measure. Show with an example that the hypothesis $\mu(X) < \infty$ is necessary.

5. On the real line with the Lebesgue σ-algebra $\mathcal{M}$, we consider the following four measures: m (Lebesgue measure), δ (the Dirac measure supported at 0), $\mu = m + \delta$ and ν, the measure defined by

$$\nu(E) = \int_E x\, dm + \delta(E), \;\; E \in \mathcal{M}.$$

   (a) Show that ν is not absolutely continuous with respect to m.

   (b) Prove that ν is absolutely continuous with respect to μ.

   (c) Find the Radon-Nikodym derivative of ν with respect to μ.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that $1 \leq p < q < \infty$. Prove that if $L^p(X)$ is not contained in $L^q(X)$, then X contains measurable sets of arbitrarily small positive measure (i.e., for every $\epsilon > 0$ there exists a set $E \in \mathcal{M}$ such that $0 < \mu(E) \leq \epsilon$).

   Hint: Consider the sets $E_n = \{x \in X : |f(x)| > n\}$, *for an appropriate function* f.