Analysis Qualifying Examination, January 2014

Choose 6 problems from Part I and 3 problems from Part II.

Part I: Real Analysis

In all problems in Part I, m denotes Lebesgue measure.

1. Show that if $E_1 \cup E_2$ is Lebesgue measurable and $m(E_2) = 0$, then E_1 is Lebesgue measurable.

2. Let
$$F(x) = \begin{cases} \arctan(x) + 5 \text{ if } x \geq 2, \\ x^2 - 2 \text{ if } 0 \leq x < 2, \\ e^x - 3 \text{ if } x < 0. \end{cases}$$
Let μ_F be the Borel measure on $\mathbb{R}$ with distribution function F.
   (a) Calculate $\mu_F(\mathbb{R})$, $\mu_F(\{2\})$, and $\mu_F((-\infty, 0))$.
   (b) Calculate the Lebesgue derivative of μ_F.
   (c) Is μ_F absolutely continuous with respect to Lebesgue measure?

3. Let $f \in L^1(X, \mu)$. Show that $\{x \in X : f(x) \neq 0\}$ is σ-finite.

4. Suppose $f \in L^1(\mathbb{R}, m)$. Show that
$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \cos(nx) dx = 0$$
(Hint: Show this first for the characteristic function of an interval.)

5. Assume that for every $\varepsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c. Show that $f_n \to f$ a.e.

6. For each $E \subseteq \mathbb{R}$, let $\mu(E) = \#(E \cap \mathbb{Z})$. Calculate
$$\iint_{[2,\infty)\times[2,\infty)} (y-1)x^{-y} d(m(x) \times \mu(y)).$$
What theorem did you use?

7. Let $F : \mathbb{R} \to \mathbb{R}$. Prove that there is a constant M such that $|F(x) - F(y)| \leq M|x-y|$ for every $x, y \in \mathbb{R}$ if and only if F is absolutely continuous and $|F'| \leq M$ almost everywhere.

8. A set $E \subseteq [0, 1]$ has the property that there exists $0 < d < 1$ such that for every $(\alpha, \beta) \subset [0, 1]$,
$$m(E \cap (\alpha, \beta)) > d(\beta - \alpha).$$
Prove that $m(E) = 1$. (Hint: Lebesgue's differentiation)

Part II: Complex and Functional Analysis

1. Use Morera's theorem to show that f defined by
$$f(z) = \int_0^\infty e^{-zt} t^{-3} \sin^3(t) dt$$
is analytic in $\Re z > 0$.

2. Find an analytic function that maps $\{z : |z| < 1, \Re z > 0\}$ onto the unit disk in a one-one fashion.

3. Show that
$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$
(Hint: Deform the contour and use $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$.)

4. Prove that a necessary and sufficient condition for a normed space X to be complete is that for every sequence of vectors $\{x_n\} \subset X$ such that
$\displaystyle\sum_{n=1}^{\infty} \|x_n\| < \infty$, the series $\displaystyle\sum_{n=1}^{\infty} x_n$ converges to an element $x \in X$.

5. Let X be a Banach space.
   (a) Define what it means for X to be separable.
   (b) and assume that X^* is separable. Prove that X is separable.

6. In the space of sequences ℓ^p, let $e_i = (0, \ldots, 0, 1, 0, 0, \ldots)$, where the 1 is in the i-th coordinate. Let p' be the conjugate exponent of p. Find a explicit sequence $b = (b_1, b_2, b_3, \ldots) \in \ell^{p'}$ such that $\|b\|_{\ell^{p'}} = 1$ and
$$\|e_1 + e_2\|_{\ell^p} = \sum_{i=1}^{\infty} (e_1 + e_2)_i b_i.$$
What theorem guarantees that such sequence b exists?

7. Use the Closed Graph Theorem to prove the following:
   Let X, Y be Banach spaces, and let $T : X \to Y$ be a linear operator such that for every $f \in Y^*$, $f \circ T \in X^*$. Show that T is continuous.