Analysis Preliminary Exam
June 7, 2010

1. Let μ^* be an outer measure on X, and $A, B \subseteq X$ such that one of them is μ^*-measurable while the other might not be μ^*-measurable. Show that

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B).$$

2. Show that on the real line there are $2^{\mathfrak{c}}$ Lebesgue measurable sets.

3. (a) State Egoroff's Theorem.
(b) Does Egoroff's Theorem hold for infinite dimensional measure spaces? If yes, justify it. If not, provide a counterexample.

4. Let $(X, \mathcal{F}, \mu)$ be an arbitrary measure space and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions such that $f_n \to f$ uniformly on X. Show that f is integrable and

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function and let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Show that $g \circ f$ is absolutely continuous.

6. Let $(X, \mathcal{F}, \mu)$ be an arbitrary measure space, $\alpha \in (0, 1)$, and let $1 < p < q < r < \infty$ such that

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{r}.$$

Show that $||f||_q \leq ||f||_p^{\alpha} ||f||_r^{1-\alpha}$, for all $f \in L^r(X)$.