Analysis Preliminary Examination
January 2013

- Unless a problem states otherwise m will denote Lebesgue measure, m^* will denote Lebesgue outer measure, and $\mathcal{L}$ will denote the Lebesgue measurable sets. You may use without proof the Lebesgue dominated convergence theorem and the Fubini-Tonelli Theorem.
- Please justify your answers.
- Clearly indicate which problems you wish to be considered for grading.

Part I: Measure Theory
Provide solutions to 6 of the problems below

1. (a) Define Lebesgue outer measure m^*.
   (b) Show: If $A \subseteq \mathbb{R}$ is countable, then $m^*(A) = 0$.
   (c) Is the converse true? Justify.
2. Let μ be counting measure on $[0,1]$ and let m be Lebesgue measure. Let f be the characteristic function of $\{(x,x) : 0 \leq x \leq 1\} \subset [0,1]^2$. Show that $f \notin L^1(\mu \times m)$. (Hint: Calculate the iterated integrals.)
3. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable. Are the following statements true or false? Justify your answers.
   (a) $\lim_{x\to\infty} |f(x)| = 0$.
   (b) $\lim_{n\to\infty} \int_{E_n} |f| dm = 0$ where $E_n = \{x \in \mathbb{R} : |f(x)| > n\}$.
4. Suppose that $f_n \to f$ in measure and $g_n \to g$ in measure. Show that $f_n + g_n \to f + g$ in measure. Give an example where $f_n g_n$ does not converge in measure to fg. (Use Lebesgue measure on the real line for this part.)
5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and let g be Lebesgue measurable. Show that
$$\lim_{t\to 0} \int_{[0,1]} g(x)|f(x+t) - f(x)|dx = 0.$$
6. If M is a σ-algebra show that M is either finite or uncountable.
7. Let $\{f_i\}_{i\in I}$ be a family of uniformly bounded measurable $\mathbb{R}$-valued functions. Show that if I is countable then $\sup(f_i)$ is measurable. Prove that this is not necessarily true if I is uncountable.
8. If $|f(x) - f(y)| \leq M|x - y|$ for all x, y show that f is absolutely continuous and that $|f'| \leq M$ a.e.
9. Let μ be a signed measure. Show that $f \in L^1(\mu)$ if and only if $f \in L^1(|\mu|)$.

Part II: Complex and Functional Analysis
Provide solutions to 3 of the problems below

II.1 Let f be an entire function. Prove: If $\Re f(z) \geq 0$ for all $z \in \mathbb{C}$, then F is constant.

II.2 Let f be an entire function and suppose there are constants $M > 0$ and $R > 0$, and $n \in \mathbb{N}$ such that $|f(z)| \leq M|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree at most n.

II.3 Prove
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

II.4 Let H be a Hilbert space and assume that $\varphi : H \to \mathbb{C}$ is a continuous linear functional. Prove that there is a unique $h_0 \in H$ such that $\varphi(h) = \langle h, h_0 \rangle$ for all $h \in H$.

II.5 Let $\{e_1, \cdots, e_n\}$ be an orthonormal set in a Hilbert space H and let M be subspace spanned by the orthonormal set. Prove that M is closed and that if P is the projection onto the subspace M then $Px = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$ for all $x \in H$.

II.6 Let H be a Hilbert space. We say that a sequence $\{T_n\}$ in $B(H)$ converges in the weak operator topology to $T \in B(H)$ if $\lim_{n\to\infty} \langle T_n h, k \rangle = \langle Th, k \rangle$ for all h and k. Show that if T_n converges in norm to T that T_n converges in the weak operator topology to T.