Analysis Premilinary Exam
June 2009

1. Let f_n, f be in $L^1(\mathbb{R})$. For each one of the following statements, give a proof if it is true or a counterexample if it is false.

   (a) If each function f_n is finitely supported and f_n converges to f pointwise, then f_n converges to f in $L^1(\mathbb{R})$.

   (b) If f_n converges to f in $L^1(\mathbb{R})$, then f_n converges to f in measure.

   (c) If f_n converges to f uniformly then f_n converges to f in $L^1(\mathbb{R})$.

2. Let μ be a positive measure and λ a signed measure. Assume that λ and μ are mutually singular and that λ is absolutely continuous with respect to μ. Prove that $\lambda = 0$.

3. Let $f_n, f \in L^p(X)$, $g_n, g \in L^{p'}(X)$, where p' is the conjugate exponent to p. Assume that $f_n \to f$ in $L^p(X)$ and $g_n \to g$ in $L^{p'}(X)$. Prove that $f_n g_n$ converges to fg in $L^1(X)$.

4. In $\mathbb{R}^2$ we consider the measure $\pi = dx \otimes \delta_0$, where dx is Lebesgue measure on $\mathbb{R}$ and δ_0 is the Dirac measure supported at the origin.

   (a) Let $E = \{(x, y) \in \mathbb{R}^2 : |x - y| < 1\}$. Find $\pi(E)$.

   (b) Find
   $$\int_{\mathbb{R}^2} \frac{1}{x^2 + (y-1)^2} \, d\pi(x, y).$$

5. Let $f_n : [0, 1] \to \mathbb{R}$ be absolutely continuous functions such that for every $n \in \mathbb{N}$, the total variation of f_n is at most 1. Suppose that f_n converges to f pointwise.

   (a) Must f have bounded variation on $[0, 1]$? (Use the definition of bounded variation).

   (b) Must f be absolutely continuous on $[0, 1]$?

6. Let m^* be the outer Lebesgue measure on $\mathbb{R}$. Prove that for every $E \subseteq \mathbb{R}$,
   $$m^*(E) = \sum_{n=-\infty}^{\infty} m^*(E \cap [n, n+1)).$$

   Hint: Use the definition of measurable set.