# Analysis Preliminary Examination

August 2006

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- Unless a problem states otherwise you can assume that any unspecified measure is Lebesgue measure.

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1. If $\{f_n\}$ is a sequence of measurable real-valued functions prove that $\limsup_n f_n$ is measurable.

2. (a) State what it means for a set to be measurable.

   (b) If $E_1$ and $E_2$ are measurable sets prove that $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + M(E_2)$.

3. Prove that every open set in $\mathbb{R}$ and every closed set in $\mathbb{R}$ is measurable.

4. Define
$$f(x) = \begin{cases} e^x & x \in E \\ -e^x & x \in E^c \end{cases}$$
where $E$ is a nonmeasurable subset of $\mathbb{R}$. Show that $f^{-1}(t)$ is measurable for every $t \in \mathbb{R}$. On the other hand show that $f$ is not a measurable function.

5. Prove that an algebra of sets $A$ is a $\sigma$-algebra of sets if and only if $A$ is closed under countable increasing unions (i.e. If $\{a_i\}_{i=1}^{\infty} \subseteq A$ and $a_i \subseteq a_{i+1}$ for all $i$, then $\cup a_i \in A$).

6. Let $f$ be of bounded variation on $[a, b]$. Show that
$$\int_{[a,b]} |f'| \leq T$$
where $T$ is the total variation of $f$ on the interval $[a, b]$.

7. Let $f \in L^1 \cap L^2$ and let $A = \{x : |f(x)| \geq 1\}$. Define
$$g(x) = \begin{cases} |f(x)|^2 & x \in A \\ |f(x)| & x \in A^c \end{cases}.$$
Use $g$ to show that $f \in L^p$ for all $1 \leq p \leq 2$ and that
$$\lim_{p \to 1^+} \|f\|_p = \|f\|_1.$$

8. Let $(X, \mu, \mathcal{B})$ be a measure space. Assume that $\{f_n\}$ and $\{g_n\}$ are sequences of real-valued functions on $X$ converging in measure to $f$ and $g$, respectively. Show that if $\mu(X) < \infty$ then $f_n g_n \to fg$ in measure. Show by way of counterexample that if $\mu(X) = \infty$ this is not true.

9. Let $g : X \to \mathbb{R}$ be a $\mu$-integrable function, and let $h : Y \to \mathbb{R}$ be a $\nu$-integrable function, where $\mu$ and $\nu$ are arbitrary measures on $X$ and $Y$, respectively. Define $f : X \times Y \to \mathbb{R}$ by $f(x, y) = g(x)h(y)$ for each $x, y$. Show that $f$ is $\mu \times \nu$ integrable and
$$\int f \, d(\mu \times \nu) = \left( \int_X g \, d\mu \right) \cdot \left( \int_Y h \, d\nu \right).$$

10. Let $A \subset [0, 1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for all interval $I \subseteq [0, 1]$. Let $F(x) = m([0, x] \cap A)$. Show that $F(x)$ is absolutely continuous and strictly increasing on $[0, 1]$ but $F'(x) = 0$ on a set of positive measure.