Analysis Preliminary Examination

August 2007

- Unless a problem states otherwise m will denote Lebesgue measure.
- Please justify your answers.

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1. Suppose $f \in L^1(\mathbb{R}, m)$, show that
$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\cos(nx)\, dm = 0.$$

2. Show that if $f_n \to f$ in $L^1(\mathbb{R}, m)$ then $f_n \to f$ in measure. Is the converse true?

3. Let m^* denote Lebesgue outer measure on $\mathbb{R}$ and suppose that $E \subseteq \mathbb{R}$ has the property that
$$m^*(E \cap (a,b)) < \frac{3}{4}(b-a)$$
for every finite open interval (a,b). Show that E has 0 measure.

4. If f is continuous on $[0,1]$ is it of bounded variation?

5. Decide whether the following are true or false.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable, then $\lim_{x\to\infty} |f(x)| = 0$.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable, then setting
   $$E_n := \{x \in \mathbb{R} : |f(x)| > n\}$$
   we have $\lim_{n\to\infty} \int_{E_n} |f| = 0$.

6. Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function such that $f(x,t)$ is a measurable function of x, for each $t \in \mathbb{R}$. Assume further that for each $x \in \mathbb{R}$, $f(x,t)$ is a continuous function of t. If there exists an integrable function $g : \mathbb{R} \to \mathbb{R}$ such that for each $t \in \mathbb{R}$, the inequality $|f(x,t)| \leq g(x)$ holds for almost every $x \in \mathbb{R}$ then the function
$$F(t) = \int_{\mathbb{R}} f(x,t)\, dm(x)$$
is a continuous function of t.

7. Let M be a closed subspace of the Banach space X. For $x \in X \setminus M$ prove that there exists $\varphi \in X^*$ such that $\|\varphi\| = 1, \varphi|_M = 0$ and $\varphi(x) = \inf\{\|x-y\| : y \in M\}$.

8. Let K be a compact subset of a metric space, and assume that $\{G_i\}$ is an open cover of K. Prove that there exists $\varepsilon > 0$ such that for every $x \in K$, there is an i with $B(x,\varepsilon) \subseteq G_i$.

9. Suppose that μ and ν are σ-finite measures on X with $\nu \ll \mu$ and let $\lambda = \mu + \nu$. If $f = \frac{d\nu}{d\lambda}$ then $0 \leq f \leq 1$ μ-a. e. and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

10. Consider a measure space (X,μ) with $\mu(X) = 1$, and let $f, g \in L^2(\mu)$. If $\int f\, d\mu = 0$ then use Hölder's Inequality to deduce that
$$\left(\int fg\, d\mu\right)^2 \leq \left(\int g^2\, d\mu - \left(\int g\, d\mu\right)^2\right)\int f^2\, d\mu.$$