Analysis Preliminary Examination
Spring 2012

- Unless a problem states otherwise m will denote Lebesgue measure, m^* will denote Lebesgue outer measure, and $\mathcal{L}$ will denote the Lebesgue measurable sets. You may use without proof the Lebesgue dominated convergence theorem.
- Please justify your answers.
- Clearly indicate which problems you wish to be considered for grading.

Part I: Measure Theory
Provide solutions to 6 of the problems below

I.1 Define what it means for a function $\mu : X \to [0, \infty]$ to be a measure on $(X, \mathcal{M})$. Use your definition to prove that if μ_1 and μ_2 are measures on $(X, \mathcal{M})$ then $\alpha_1\mu_1 + \alpha_2\mu_2$ is a measure on $(X, \mathcal{M})$ where $\alpha_i \in [0, \infty)$.

1.2 Prove: If f is an integrable function which is positive a.e. on a measurable set E and $\int_E f d\mu = 0$, then $\mu(E) = 0$. (Hint: Consider $F_n = \{x : f(x) \geq 1/n\}$.)

I.3 Define what it means to be a Borel measure on $\mathbb{R}$. Define a Borel measure on $\mathbb{R}$ such that $\mu(\{0\}) = 1$.

I.4 (a) Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of measurable functions. Show that

$$g(x) = \sup_j f_j$$

is a measurable function.

(b) Does the conclusion remain valid if the supremum is taken over an uncountable system of functions? Prove or give a counterexample.

I.5 State the Lebesgue Differentiation Theorem and use it to show that if E is a Borel set in $\mathbb{R}$ and we define the density of E at x to be

$$D_E(x) = \lim_{r\to\infty} \frac{m(E \cap B(x, r))}{m(B(x, r))}$$

then $D_E(x) = 1$ for almost every $x \in E$ and $D_E(x) = 0$ for almost every $x \in E^c$.

I.6 Show: If $f_n \to f$ in $L^1(\mu)$, then $f_n \to f$ in measure. Is the converse of this statement true? Prove your answer.

I.7 Show that counting measure on $\mathcal{B}_{[0,1]}$ has no Lebesgue decomposition with respect to Lebesgue measure. (Hint: Prove first that Lebesgue measure is absolutely continuous with respect to counting measure.)

I.8 Prove that any bounded increasing function on $\mathbb{R}$ is of bounded variation.

I.9 Find

$$\lim_{n\to\infty} \int_0^\infty \frac{\cos(t/n)}{e^{-nt} + e^t} dt.$$

Justify your answer.

Part II: Complex and Functional Analysis

Provide solutions to 3 of the problems below

II.1 Recall that a series $\sum x_n$ in a normed vector space is absolutely convergent if $\sum \|x_n\|$ converges in $\mathbb{R}$. Prove that a normed vector space is complete if and only if every absolutely convergent series in X converges in X.

II.2 Prove that a linear operator on a Banach space X is continuous if and only if it is bounded.

II.3 Consider the Hilbert space $\ell^2(\mathbb{N})$ and consider the closed subspace M spanned by elements of the form $\{(1, 1, 1, \cdots, 1, 0, 0, 0, \cdots)\}$. Determine $M^\perp$ and justify your answer.

II.4 (a) State Morera's theorem.

(b) Show that if $f : \mathbb{C} \to \mathbb{C}$ is continuous on $\mathbb{C}$ and analytic on $\mathbb{C}\backslash[-1, 1]$, then f is analytic on $\mathbb{C}$.

II.5 (a) State the open mapping theorem.

(b) Let f be a function that is analytic and real valued on $\{z : \Im z > 0\}$. Show that f is constant.

II.6 Give the Laurent expansion of $f(z) = 1/(z - 2)$

(a) in $\{z : |z| < 2\}$,

(b) in $\{z : |z| > 2\}$.