Analysis Preliminary Examination
January 2012

- Unless a problem states otherwise m will denote Lebesgue measure, m^* will denote Lebesgue outer measure, and $\mathcal{L}$ will denote the Lebesgue measurable sets.
- Please justify your answers.

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1. Recall that the Borel σ-algebra on $\mathbb{R}$ is the σ-algebra generated by the open sets in $\mathbb{R}$. Show that the Borel σ-algebra is also generated by
$$\{[a, b) : -\infty < a < b < \infty\}.$$
2. Let (X, M, μ) be a measure space.
   (a) Define what it means for (X, M, μ) to be complete.
   (b) Show that $(\mathbb{R}, \mathcal{L}, m)$ is complete.
3. Let $f : \mathbb{R} \to \mathbb{R}$.
   (a) Show that if f is monotonic then f is measurable.
   (b) Explain why if f is of bounded variation then f is measurable.
4. Assume f is a positive valued measurable function and define $\lambda(E) = \int_E f\, dm$ for every measurable set E.
   (a) Show that λ is a measure.
   (b) Show that for any positive valued measurable function g, we have $\int g\, d\lambda = \int fg\, dm$ (Hint: First assume that g is simple).
5. Assume that f is a measurable function on $[0, 1]$.
   (a) Give the definition of $L^p[0, 1]$ for $1 \le p \le \infty$.
   (b) Recall that if $f \in L^\infty[0, 1]$ then $f \in L^p[0, 1]$ for all $1 \le p < \infty$. Show that if $f \in L^\infty$ then
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$
6. (a) State Fatou's Lemma and show by an example that the inequality can be strict.
   (b) State the Monotone Convergence Theorem. Does the Monotone Convergence Theorem follow from Fatou's Lemma (you need not explain your answer)?
7. Let $f : [0, 1] \to \mathbb{R}$ be a function.
   (a) Show that if f is measurable then so is $|f|$.
   (b) Show that if $f \in L^p[0, 1]$ and $\varepsilon > 0$ then
$$m^*(\{x : |f(x)| > \varepsilon\}) \le \frac{\|f\|_p^p}{\varepsilon^p}$$
   for $1 \le p < \infty$.
8. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces.
   (a) State the Fubini-Tonelli Theorem(s) for $X \times Y$.
   (b) If $f(x, y) = ye^{-1(1+x^2)y^2}$ for all $(x, y) \in [0, \infty) \times [0, \infty)$, explain why
$$\int_{[0,\infty)} \left[\int_{[0,\infty)} f(x, y)\, dm(x)\right] dm(y) = \int_{[0,\infty)} \left[\int_{[0,\infty)} f(x, y)\, dm(y)\right] dm(x).$$
   (c) Use the previous part to conclude that $\displaystyle\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.