Analysis Preliminary Examination

January 2008

- Unless a problem states otherwise m will denote Lebesgue measure.
- Please justify your answers.

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1. (a) State the Lebesgue Differentiation Theorem.

   (b) Foe a Borel set $E \subseteq \mathbb{R}$ define the density of E at x to be equal to

   $$D_e(x) := \lim_{r \to 0} \frac{m(E \cap (x-r, x+r))}{2r}.$$

   Show that $D_e(x) = 1$ for almost every $x \in E$ and $D_e(x) = 0$ for almost every $x \in E^c$.

2. True or false? An open dense set in $\mathbb{R}$ must have infinite measure. Justify!

3. If $\{f_n\}$ is a sequence of real-valued measurable functions, prove that $\limsup_n f_n$ is measurable as well.

4. Let $f \in L^1(\mathbb{R}, m)$.

   (a) Show that the set $\{x : f(x) \neq 0\}$ is σ-finite.

   (b) For every $\varepsilon > 0$ there must exist a bounded function $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that

   $$m(\{x : f(x) \neq g_\varepsilon(x)\}) < \varepsilon.$$

5. If $f \in L^1(\mathbb{R}, m)$ and we define

   $$F(x) = \int_{[-\infty, x]} f \, dm$$

   then show that $F(x)$ is continuous.

6. Let X be a normed vector space:

   (a) Define what it means for a series in X to be absolutely convergent.

   (b) Show that X is complete iff every absolutely convergent series in X converges.

7. A function $F : (a, b) \to \mathbb{R}$ is called convex if

   $$F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda)F(t),$$

   for all $s, t \in (a, b)$ and $\lambda \in (0, 1)$. Show that the following are equivalent

   (a) F is convex.

   (b) For all $s, t, s', t' \in (a, b)$ with $s \leq s' < t'$ and $s < t \leq t'$

   $$\frac{F(t) - F(s)}{t - s} \leq \frac{F(t') - F(s')}{t' - s'}.$$

   (c) F is absolutely continuous on every compact subinterval of (a, b) and F' is increasing on the set where it is defined.

8. Let $\{r_1, r_2, ...\}$ be some fixed enumeration of $\mathbb{Q}$. Define a function f by

   $$f(x) := \sum_{r_j < x} 3^{-j}.$$

   Show that $0 < f(x) < 1/2$, and that f is increasing on $\mathbb{R}$. Is f absolutely continuous on $\mathbb{R}$? Justify your answer!