Analysis Preliminary Examination

May 2007

- Unless a problem states otherwise you can assume that μ is an arbitrary measure.

---

1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space.
   (a) If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$ then $\mu(E) = \mu(F)$.
   (b) Say that $E \sim F$ if $\mu(E \triangle F) = 0$. Show that $\sim$ is an equivalence relation on $\mathcal{M}$.
   (c) For $E, F \in \mathcal{M}$ define $\rho(E, F) = \mu(E \triangle F)$. Show that $\rho(E, F) \leq \rho(E, G) + \rho(G, F)$ and hence ρ defines a metric on equivalence classes of M under the relation $\sim$.

2. Suppose $\{f_n\} \subseteq L^1(\mu)$ and $f_n \to f$ uniformly.
   (a) If $\mu(X) < \infty$ then $f \in L^1(\mu)$ and $\int f_n \to \int f$.
   (b) Show by example that the conclusions of (a) can fail if $\mu(X) = \infty$.

3. If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, $f \in L^1(\mu)$, and $g \in L^1(\nu)$ define $h(x, y) = f(x)g(y)$.
   (a) Show that $\int h\, d(\mu \times \nu) = \left(\int f\, d\mu\right) \left(\int g\, d\nu\right)$.
   (b) if $\mu_1 \ll \mu$ and $\nu_1 \ll \nu$ show that
   $$\frac{d(\mu_1 \times \nu_1)}{d(\mu \times \nu)} = \frac{d\mu_1}{d\mu}\frac{d\nu_1}{d\nu}.$$

4. Let μ be counting measure on $\mathbb{N}$. Show that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly.

5. Show that a linear functional on a normed vector space is continuous if and only if it is bounded.

6. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $K : X \times Y \to \mathbb{C}$ be $\mathcal{M} \otimes \mathcal{N}$-measurable. Suppose that there exists $C > 0$ such that $\int |K(x, y)|\, d\mu(x) \leq C$ for ν-a.e. $y \in Y$. If $f \in L^1(\nu)$ the integral
   $$Tf(x) = \int K(x, y) f(y)\, d\nu(y)$$
   defines a linear map from L^1 into L^1 such that $\|Tf\|_1 \leq C\|f\|_1$.

7. Let F be absolutely continuous on $[a, b]$. Show that f is of bounded variation on $[a, b]$.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space and let L^+ denote the set of all positive valued measurable functions on X. Assuming that $f \in L^+$, show that
   $$\nu(E) = \int_E f\, d\mu$$
   defines a measure on X and further if $g \in L^+$,
   $$\int g\, d\nu = \int fg\, d\mu.$$