Analysis Preliminary Examination

MAY 2006

- Unless a problem states otherwise you can assume that any unspecified measure is Lebesgue measure.

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1. Let $\mathcal{S}$ be the collection of all subsets of $[0,1)$ which can be written as a finite union of intervals of the form $[a,b) \subseteq [0,1)$. Show that $\mathcal{S}$ is an algebra of sets, but is not a σ-algebra.

2. Let m^* denote Lebesgue outer measure on $\mathbb{R}$. Prove that m^* is countably subadditive. Use this to prove that $m^*(\mathbb{Q}) = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that the function f' is measurable.

4. Let A be a measurable subset of $\mathbb{R}$ with $0 < m(A) < \infty$. Show that for $1 < p < q \leq \infty$ we have $L^q(A) \subseteq L^p(A)$.

5. Let M denote the set of measurable functions on $[0,1]$. Given functions, $f, g \in M$ define

$$d(f,g) := \int_{[0,1]} \frac{|f-g|}{1+|f-g|}\, dm.$$

This defines a metric on M. Show that a sequence of measurable functions f_n converges to f in measure if and only if $\lim_{n\to\infty} d(f_n, f) = 0$.

6. Let $\{q_1, q_2, q_3, \cdots\}$ be some fixed enumeration of the rational numbers in $\mathbb{R}$. Define the function

$$f(x) = \sum_{q_j < x} 3^{-j}.$$

Show that $0 < f(x) < \frac{1}{2}$ and $f(x)$ is increasing on $\mathbb{R}$. Is f absolutely continuous on $\mathbb{R}$? (Be sure to justify your answer).

7. Show that if f and g are absolutely continuous on $[a,b]$ then $f \cdot g$ is absolutely continuous on $[a,b]$.

8. Prove that if $A \subseteq \mathbb{R}$ and every subset of A is measurable then $m(A) = 0$.

9. Let μ denote counting measure on $\mathbb{N}$. Is the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ σ-finite, where $\mathcal{P}(\mathbb{N})$ is the σ-algebra of subsets of $\mathcal{N}$? Is this measure space complete? (Be sure to justify your answers)

10. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(x,t)$ is a measurable function of x, for each $t \in \mathbb{R}$. Assume further that for each $x \in \mathbb{R}$, $f(x,t)$ is a continuous function of t. If there exists an integrable function $g : \mathbb{R} \to \mathbb{R}$ such that for each $t \in \mathbb{R}$, the inequality $|f(x,t)| \leq g(x)$ holds for almost every $x \in \mathbb{R}$ then the function

$$F(t) = \int_{\mathbb{R}} f(x,t)\, dm(x)$$

is a continuous function of t.