Department of Mathematics - North Dakota State University
Applied Mathematics Qualifying Exam
MAY 2008
Subject: MATH 784 Partial Differential Equations I

1. (25 points) Let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded domain, and let $u : \Omega \to \mathbb{R}$ be a harmonic function.
(i) Prove that

$$|\nabla u(x)| \leq \frac{N}{\text{dist}(x, \partial\Omega)} \sup_{\Omega} |u|,$$

for all $x \in \Omega$.
(ii) If, in addition, $m \leq u(x) \leq M$ for all $x \in \Omega$, then prove that

$$|\nabla u(x)| \leq \frac{N}{2\ \text{dist}(x, \partial\Omega)}(M - m).$$

2. (25 points) Let

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} < 1 \right\},$$

and assume that $u \in C^2(\overline{\Omega})$ is a solution of the problem

$$\begin{cases} \triangle u + 1 = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that

$$\frac{2}{3} \leq u(0, 0, 0) \leq \frac{25}{6}.$$

3. (25 points) Use the method of characteristics to solve the problem

$$\begin{cases} u_x^2 + y u_y - u = 0 \\ u(x, 1) = \frac{x^2}{4} + 1 \ (x \in \mathbb{R}). \end{cases}$$

4. (25 points) Let $\Omega \subset \mathbb{R}^N$ be a connected, bounded, open set with smooth boundary. Let $f : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ be a smooth function such that there exists $M > 0$ for which

$$\int_{\Omega} f^2(x, t) dx \leq M^2 \text{ for all } t \geq 0.$$

Let $u \in C^2(\overline{\Omega} \times [0, \infty))$ be the solution of the nonhomogeneous heat equation

$$\begin{cases} u_t - \triangle u = f & \text{in } \Omega \times (0, \infty) \\ u(\cdot, 0) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times [0, \infty). \end{cases}$$

(i) Show that there exists a constant $C > 0$ such that

$$\int_{\Omega} u^2(x,t)dx < C \text{ for all } t \geq 0.$$

(ii) Would the statement in part (i) still be true if the Dirichlet boundary conditions were replaced by the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0, \infty) \text{ ?}$$

If yes, prove your claim. If no, provide a counterexample.