Department of Mathematics - North Dakota State University
Applied Mathematics Qualifying Exam
MAY 2008
Subject: MATH 785 Partial Differential Equations II

1. (25 points) Let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded domain, and let $f \in L^2(\Omega)$ be such that $f \leq 0$ a.e. in Ω. If $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing and $g(0) = 0$, show that the weak solution of the problem

$$\begin{cases} -\triangle u + g(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

is such that $u \leq 0$ a.e. in Ω.

2. (25 points) Let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded domain which satisfies the interior ball condition at every point on its boundary, and let $f, \varphi, c, \alpha : \overline{\Omega} \to \mathbb{R}$ be continuous functions such that $c(x) \leq 0$ for all $x \in \Omega$ and $\alpha(x) \geq 0$ for all $x \in \partial\Omega$. Let $v, w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be two solutions of the Robin problem

$$\begin{cases} \triangle u + cu = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \alpha u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Prove that $v - w$ is a constant function in Ω. If $\alpha > 0$ in Ω deduce that the above problem has at most one solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$.

3. (25 points) Let $\Omega \subseteq \mathbb{R}^N$ be an open, bounded, convex domain containing the origin, and assume that $\partial\Omega \in C^2$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and define $F(z) := \int_0^z f(t)dt$.

(i) Explain (but do not proceed with the lengthy computations) how you would prove that any solution $u \in C^2(\overline{\Omega})$ of

$$\begin{cases} -\triangle u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

satisfies Pohožaev's identity:

$$\left(\frac{N}{2} - 1\right) \int_\Omega |Du|^2 dx + \frac{1}{2} \int_{\partial\Omega} (\nu \cdot x) \left(\frac{\partial u}{\partial \nu}\right)^2 dS = N \int_\Omega F(u) dx$$

(ii) Use the result in part (i) to prove that any eigenfunction φ of the operator $-\triangle$ with Dirichlet boundary conditions satisfies $\frac{\partial \varphi}{\partial \nu} \neq 0$.

4. (25 points) Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain, and let $f \in L^2(\Omega)$ be given. For $\varepsilon > 0$ define

$$\beta_\varepsilon(y) = \begin{cases} 0 & \text{if } y \geq 0, \\ \frac{y}{\varepsilon} & \text{if } y \leq 0, \end{cases}$$

and let $u_\varepsilon \in H^1_0(\Omega)$ be the weak solution of the problem

$$\begin{cases} -\triangle u_\varepsilon + \beta_\varepsilon(u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that as $\varepsilon \to 0^+$ we have that $u_\varepsilon \rightharpoonup u$ weakly in $H^1_0(\Omega)$, where u is the unique solution of the variational inequality

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \int_\Omega f(v-u)dx$$

for all $v \in H^1_0(\Omega)$, with $v \geq 0$ a.e. in Ω.