Preliminary Exam
Applied Mathematics
Summer 2012

1. (*7 points*) Solve the equation

$$y' = (4t + y - 3)^2.$$

2. (*10 points*) Consider an ordinary differential equation $y' + a(t)y = f(t)$ with continuos $a(t) \geq c > 0$ for all t. Let $y_0(\cdot)$ be a solution with the initial data $y_0(0) = b$. Show that for every positive number ϵ there exists a positive number δ such that for all functions $f_1(\cdot)$ and numbers b_1 satisfying $|f_1(t) - f(t)| < \delta\ \forall t > 0$, $|b - b_1| < \delta$ for a solution $y_1(\cdot)$ of the initial value problem $y' + a(t)y = f_1(t)$, $y(0) = b_1$ we have $|y(t) - y_1(t)| < \epsilon\ \forall t > 0$.

3. (*13 points*) Show that there is only one solution of the equation $ty' - (3t^2 + 2)y = t^3$ which has a finite limit at $+\infty$. Find this limit.

4. (*8 points*) For which n the existence of a solution $y = x(e^x - 1)$ of a linear differential equation $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_0(t)y = 0$ does not contradict the Picard theorem about existence and uniqueness of solutions of initial value problems?

5. (*12 points*) Using Lyapunov functions check if the following system is globally asymptotically stable

$$\begin{aligned} \dot{x} &= y - 3x - x^3, \\ \dot{y} &= 6x - 2y. \end{aligned}$$

6. (*13 points*) Assume function p and q are continuous in $[a, b]$ and a linear differential equation $y'' + p(t)y' + q(t)y = 0$ has no nonzero solutions y such that $y(a) = 0$, $y(b) = 0$. Show that for every numbers c, d there exists a unique solution of the boundary value problem with $y(a) = c$, $y(b) = d$.

7. (*10 points*) Find all eigenvalues and eigenfunctions for the following problem

$$t^2y'' + 3ty' = \lambda y, \quad y(1) = 0, \quad y(a) = 0 \ (a > 1).$$

8. (*11 points*) Consider a solution of differential equation

$$y' = e^{-y}\sin(e^y)$$

with an initial condition $y(0) = 0$. Can it be extended to the whole axis $-\infty < x < \infty$?

9. (*8 points*) Assume $P_n^{(\alpha,\beta)}$ is a polynomials of degree n such that $\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \delta_{nm}$ (that is, $\{P_n^{(\alpha,\beta)}\}$ are the Jacobi polynomials), $\alpha > -1$, $\beta > -1$. Show that $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\alpha,\beta)}(-x)$ for all $n = 0, 1, 2, \ldots$ and x.

10. (*8 points*) Reduce the Model Matching problem $\|T_1 - T_2QT_3\|_\infty \to \min$ over $Q \in H_\infty$ to the Nehari problem and solve it for

$$T_1(s) = \frac{1}{s+1}, \quad T_2(s) = \frac{s+2}{s+3}, \quad T_3(s) = \frac{s-1}{s+3}.$$