Preliminary Exam
Applied Mathematics
Summer 2012

1. (*7 points*) Solve the equation

$$2y' - \frac{t}{y} = \frac{ty}{t^2 - 1}.$$

2. (*12 points*) Consider equation $ty' + ay = f(t)$, where $a = const < 0$, and $f(t) \to b$ as $t \to 0$. Show that all solutions of this equation have the same limit at zero. Find this limit.

3. (*14 points*) Show that there is only one solution of the equation $ty' - (2t^2 + 1)y = t^2$ which has a finite limit at $+\infty$. Find this limit.

4. (*10 points*) Assume function $f(t, y)$ is continuous in t, y, and for every t it is non increasing with respect to y. Prove that if two solutions of equation $y' = f(t, y)$ have the same initial condition $y(t_0) = y_0$, then they coincide for all $t \geq t_0$.

5. (*11 points*) Using Lyapunov functions check if the following system is globally asymptotically stable

$$\begin{aligned} \dot{x} &= x - y - xy^2, \\ \dot{y} &= 2x - y - y^3. \end{aligned}$$

6. (*7 points*) Prove that in case $q(t) > 0 \ \forall t$ for every solution of equation $y'' + q(t)y = 0$ the quotient $y'(t)/y(t)$ is decreasing as t increases on every interval where $y(t) \neq 0$.

7. (*10 points*) Find all eigenvalues and eigenfunctions of the following problem

$$t^2 y'' + ty' = \lambda y, \quad y(1) = 0, \quad y(a) = 0 \ (a > 1).$$

8. (*10 points*) Find upper and lower bounds of solutions of the ODE

$$t^2 y'' + ty' - 4y = f(t)$$

with boundary conditions: $y(t)$ is bounded as $t \to 0$ and as $t \to \infty$, provided that $0 \leq f(t) \leq m$ for all $t \in [0, \infty)$.

9. (*11 points*) Assume $\{p_n\}$ are classical orthogonal polynomials on an interval $[a, b]$: for a smooth function ρ positive in (a, b) we have $\int_a^b \rho(x) p_n(x) p_m(x) dx = \delta_{nm}$ for all natural $n, m = 0, 1, 2, \ldots$. Denote $p_n(x) = a_n x^n + b_n x^{n-1} + \ldots$. Prove the identity

$$\sum_{k=0}^{n} p_k(x)^2 = \frac{a_n}{a_{n+1}} [p_n(x) p'_{n+1}(x) - p_{n+1}(x) p'_n(x)].$$

10. (*8 points*) Reduce the Model Matching problem $\|T_1 - T_2 Q T_3\|_\infty \to \min$ over $Q \in H_\infty$ to the Nehari problem and solve it for

$$T_1(s) = \frac{1}{s+2}, \quad T_2(s) = \frac{s-1}{s+2}, \quad T_3(s) = \frac{s+2}{s+3}.$$