Preliminary Exam
Applied Mathematics
May 20, 2013

Name: ______________________

Answer any ten problems. Write all your work and final answers on blank paper. Write your name on each piece of paper and include this exam sheet on top.

1. (10 points) Let B_t be a 2-dimensional Brownian motion and put

$$D_r = \{x \in \mathbb{R}^2; |x| < r\} \quad \text{for} \quad r > 0.$$

Compute $P^0[B_t \in D_r]$.

2. (10 points) Let $X : \Omega \to \mathbb{R}^n$ be a random variable such that $E[|X|^p] < \infty$, for some $0 < p < \infty$. Prove that

$$P[|X| \geq \lambda] \leq \frac{1}{\lambda^p} E[|X|^p], \quad \text{for all} \quad \lambda \geq 0.$$

3. (10 points) The random process $Z(t)$ is defined as $Z(t) = \alpha B(t) - \sqrt{\beta} B^*(t)$, where B and B^* are independent standard one-dimensional Brownian motions, and α and β are arbitrary positive constants. Determine the relationship between α and β for which $Z(t)$ is a Brownian motion.

4. (10 points) Define martingale. Prove that $N_t = B_t^3 - 3tB_t$ is a martingale.

5. (10 points) If M is a martingale (with respect to filtration $\mathcal{F}$), show that

$$E[(M(u) - M(s))^2 | \mathcal{F}(s)] = E[M(u)^2 - M(s)^2 | \mathcal{F}(s)].$$

6. (10 points) Let $B_t \in \mathbb{R}$, $B_0 = 0$. Define

$$\beta_k(t) = E[B_t^k]; \quad k = 0, 1, 2, \ldots; t \geq 0.$$

Use Itô's formula to prove that

$$\beta_k(t) = \frac{1}{2} k(k-1) \int_0^t \beta_{k-2}(s)\, ds, \quad k \geq 2.$$

7. (10 points)

(a) Let Y be a real valued random variable on $(\Omega, \mathcal{F}, P)$ such that $E[|Y|] < \infty$. Define $M_t = E[Y|\mathcal{F}_t]$, $t \geq 0$. Show that M_t is an $\mathcal{F}_t$ martingale.

(b) Martingale representation Theorem confirms the existence of appropriate g such that,

$$M_t = E[M_0] + \int_0^t g(s, \omega)\, dB(s), \quad t \in [0, T].$$

Find such g for M_t in part (a) where $Y = B^2(T)$.

8. (10 points)

(a) Solve the *Ornstein-Uhlenbeck* equation

$$dX_t = \mu X_t dt + \sigma dB_t,$$

where μ, σ are real constants and $B_t \in \mathbb{R}$.

(b) Find $E[X_t]$ and $Var[X_t]$ for *Ornstein-Uhlenbeck process.*

9. (10 points)

(a) Define local martingale with respect to a given filtration $\{\mathcal{N}_t\}$. Show that if $Z(t)$ is a local martingale and there exists a constant $T < \infty$ such that the family $\{Z(\tau)\}_{\tau \le T}$ is uniformly integrable then $\{Z(t)\}_{t \le T}$ is a martingale.

(b) Show that if $Z(t)$ is a lower bounded local martingale, then $Z(t)$ is a supermartingale.

10. (10 points) Let B_t be one-dimensional and let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Prove that if $t < T$ then

$$E^x[f(B_T)|\mathcal{F}_t] = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} f(x) \exp\left(-\frac{(x - B_t(\omega))^2}{2(T-t)}\right) dx.$$

11. (10 points) If $B(t)$ is one-dimensional Brownian motion and $0 < T < \infty$, derive the variance of

$$TB(T) - \int_{t=0}^{T} B(t)\, dt$$

and

$$\int_{t=0}^{T} \sqrt{|B(t)|}\, dB(t).$$