# Problems for Preliminary Exam
# Applied Mathematics
# June 2014

**Answer any eight (8) problems. All problems have 10 points.**

**1.** Assume $u$ is a solution of the heat equation $u_t - \Delta u = 0$ in $U_T = U \times (0, T]$, where $U$ is an open bounded set. Prove that function $v = \|Du\|^2 + u_t^2$ satisfies inequality $v_t - \Delta v \leq 0$ in $U_T$.

**2.** Show that general solution of equation

$$u_{xx} + u_{yy} - 2\lambda u_x - 2\mu u_y + (\lambda^2 + \mu^2)u = 0$$

with constants $\lambda$ and $\mu$ may be given by the formula

$$u(x, y) = e^{\lambda x + \mu y} v(x, y),$$

where $v$ is an arbitrary harmonic function.

**3.** Use separation of variables to solve the boundary value problem

$$\begin{gathered} u_t - a^2 u_{xx} = 0, \quad 0 < x < l,\ t > 0, \\ u_x(0, t) = u_x(l, t) = 0, \quad t > 0, \\ u(x, 0) = Bx, \quad 0 < x < l. \end{gathered}$$

**4.** Solve the following boundary value problem by characteristics.

$$u_t + u_x = u, \ \ u(x, 0) = g(x), \quad x \in R,\ t > 0.$$

**5.** Let $U = B(0, 1)$ be the open ball in $R^n$. Show that a "typical" function $u \in L^p(U)$ $(1 \leq p < \infty)$ does not have a trace on $\partial U$. That is, prove that there does not exist a bounded linear operator $T : L^p(U) \to L^p(\partial U)$ such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$.

**6.** For a two dimensional Riemannian space with metric

$$ds^2 = \frac{-a^2\, dr^2}{(r^2 - a^2)^2} + \frac{r^2\, d\theta^2}{r^2 - a^2}, \quad r > a,$$

show that the differential equation of the geodesic is

$$a^2 \left(\frac{dr}{d\theta}\right)^2 + a^2 r^2 = k^2 r^4,$$

where $k^2$ is a constant.

**7.** For a *flat* Riemannian space suppose the metric is given by

$$ds^2 = f(r)[(dx^1)^2 + (dx^2)^2],$$

where $r^2 = (x^1)^2 + (x^2)^2$. Show that $f(r) = Ar^k$, where $A$ and $k$ are constants.

**8.** Is it always possible to synchronize clocks along closed paths for a Riemannian space? If *yes* justify your answer. If *not* derive the necessary conditions that are required.

**9.** Consider the motion (not necessaroly geodesic) of a massive object inside a Schwarzschild black hole, $r < 2GM$. Use the ordinary Schwarzschild coordinates $(t, r, \theta, \phi)$. Show that $r$ must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}.$$

Calculate the maximum proper time for a trajectory from $r = 2GM$ to $r = 0$.