Problems for Preliminary Exam
Applied Mathematics
June 2015

Part I
Answer any FIVE (5) problems from Part I. All problems have 10 points.

1. Suppose that $u(x,t)$ solves

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g, \quad u_t = h, & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

where g and h are smooth and have compact support. Show that there exists a constant λ such that

$$|u(x,t)| \leq \frac{\lambda}{t},$$

for $x \in \mathbb{R}^3$ and $t > 0$.

2. Verify that if $n > 1$, the unbounded function $u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, where U is an open unit ball in $\mathbb{R}^n$ with center at the origin.

3. Assume u is a smooth solution of

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = f$$

in U, and $u = 0$ on ∂U. Assume f is bounded and L is uniformly elliptic with smooth coefficients. Fix $x_0 \in \partial U$. A *barrier* at x_0 is a C^2 function w such that

$$Lw \geq 1 \text{ in } U, \quad w(x_0) = 0, \quad w \geq 0 \text{ on } \partial U.$$

Show that if w is a barrier at x_0, there exists a constant C such that

$$|Du(x_0)| \leq C\left|\frac{\partial w}{\partial \nu}(x_0)\right|,$$

where ν is the outer unit normal to U.

4. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g, & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where g and h are smooth functions and $c \in \mathbb{R}$.

5. Assume u is a harmonic function in an open bounded set U. Prove that the function $v = \|Du\|^2$ satisfies inequality $-\Delta v \leq 0$ in U.

6. Write down an explicit formula for a solution of

$$\begin{cases} u_t + b \cdot Du + cu = 0, & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g, & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where g is a smooth functions and $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Part II
Answer any FIVE (5) problems from Part II. All problems have 10 points.

1. For which a each solution to

$$\dot{x} = |x|^a$$

is defined globally (i.e., for all $t \in \mathbb{R}$)?

2. Suppose that the linear operator $\boldsymbol{A}\colon \mathbb{R}^k \longrightarrow \mathbb{R}^k$ has a real eigenvalue $\lambda < 0$. Show that the equation $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ has at least one nontrivial solution $t \mapsto \boldsymbol{x}(t)$ such that

$$\lim_{t \to \infty} \boldsymbol{x}(t) = 0.$$

3. For which $a \in \mathbb{C}$ and $b \in \mathbb{C}$ all the solutions to

$$\ddot{x} + a\dot{x} + bx = 0$$

are bounded for all $t \in \mathbb{R}$?

4. Determine the stability properties of the equilibrium $(x, \dot{x}) = (0, 0)$ for the equation

$$\ddot{x} + x^n = 0, \quad n \in \mathbb{N} = \{1, 2, \ldots\}.$$

5. For which α the system

$$\dot{x}_1 = x_2 + \alpha x_1 - x_1^5, \quad \dot{x}_2 = -x_1 - x_2^5$$

has a stable equilibrium $(x_1, x_2) = (0, 0)$?

6. Find the eigenvalues and eigenvectors for the linear operator

$$L := -\frac{d^2}{dx^2}$$

with the boundary conditions $y(0) = y'(1) = 0$. Find Green's function for this operator.