Problems for Preliminary Exam
Applied Mathematics
September 2015

Part I
All problems have 10 points.

1. Let in the equation

$$\dot{x} + a(t)x = f(t)$$

$a(t) \geq C > 0$, $f(t) \to 0$ as $t \to \infty$, and functions a and f are continuous. Prove that each solution to this equation approaches zero as $t \to \infty$.

2. For which a each solution to

$$\dot{x} = (x^2 + e^t)^a$$

can be continued to the whole real line $t \in \mathbf{R}$?

3. Does there exist a real 2×2 matrix $\boldsymbol{S}$ such that

$$e^{\boldsymbol{S}} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}?$$

4. Investigate the stability of the trivial equilibrium of

$$\dot{x} = -2y - x^3,$$
$$\dot{y} = 3x - 4y^3.$$

5. Find Green's function for the linear differential operator

$$L = \frac{d^2}{dx^2} + 1, \quad y(0) = y(\pi), \quad y'(0) = y'(\pi).$$

Part II
All problems have 10 points.

1. Let U be bounded with a C^1 boundary. Show that a "typical" function $u \in L^p(U)$, $1 \leq p < \infty$ does not have a trace on ∂U. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\overline{U}) \cap L^p(U)$.

2. Let u be a smooth solution of

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = 0$$

in U, where L is uniformly elliptic with smooth coefficients. Define $v := |Du|^2 + \lambda u^2$. Show that $Lv \leq 0$ in U, if λ is large enough. Deduce the following:

$$||Du||_{L^\infty(U)} \leq C(||Du||_{L^\infty(\partial U)} + ||u||_{L^\infty(\partial U)}),$$

where C is some constant.

3. Let

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} + cu,$$

where L is uniformly elliptic with smooth coefficients. Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot, \cdot]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided $c(x) \geq -\mu$, for $x \in U$.

4. Prove that there exists a constant C, depending only on n (dimension of the space), such that

$$\max_{B(0,1)} |u| \leq C(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|),$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f, & \text{in } B^0(0,1) \\ u = g, & \text{on } \partial B(0,1), \end{cases}$$

where $B^0(0,1)$ and $B(0,1)$ are open and closed unit ball respectively with center at the origin of $\mathbb{R}^n$.

5. Assume u is a solution of the heat equation $u_t - \Delta u = 0$ in $U_T = U \times (0, T]$, where U is an open bounded set. Prove that function $v = \|Du\|^2 + u_t^2$ satisfies inequality $v_t - \Delta v \leq 0$ in U_T.