Problems for Preliminary Exam
Applied Mathematics
January 2016

Part I
All problems have 10 points.

1. Sketch several integral curves of the equation

$$\dot{x} = \frac{x-t}{x+t}, \quad x(t) \in \mathbf{R}.$$

2. Prove that the equation

$$\dot{x} = 2\sqrt[3]{tx}, \quad x(t) \in \mathbf{R},$$

has more than one solution passing through the origin.

3. For which matrices $\boldsymbol{A}$ each solution to the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(t) \in \mathbf{R}^k$$

is bounded for $-\infty < t < \infty$?

4. Determine the stability properties of the origin for the system

$$\dot{x} = ax + y + (a+1)x^2,$$
$$\dot{y} = x + ay,$$

where a is a real parameter.

5. Find Green's function for the linear differential operator

$$L := -\frac{d^2}{dx^2}, \quad u(0) = u'(1) = 0.$$

Write down the solution to the equation

$$-u'' = 1, \quad u(0) = u'(1) = 0$$

using the found Green's function.

Part II
All problems have 10 points.

1. Show that Laplace equation is invariant under rotation. Does that mean all harmonic functions are radial? Give counterexample if necessary.

2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

3. Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0, \quad \text{a.e. in} \quad U.$$

Prove that u is constant a.e. in U.

4. Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ solve Maxwell's equations $\mathbf{E}_t = \text{curl}\mathbf{B}$, $\mathbf{B}_t = -\text{curl}\mathbf{E}$, $\text{div}\mathbf{B} = \text{div}\mathbf{E} = 0$. Show that $u_{tt} - \Delta u = 0$, where $u = E^i$ or B^i, $i = 1, 2, 3$.

5. A function $u \in H_0^2(U)$ is a weak solution of the problem:

$$\begin{aligned} \Delta^2 u = f, \quad &\text{in} \quad U, \\ u = \frac{\partial u}{\partial \nu} = 0, \quad &\text{on} \quad \partial U, \end{aligned} \tag{1}$$

provided

$$\int_U \Delta u \Delta v \, dx = \int_U f v \, dx,$$

for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove that there exists a unique weak solution of (1).