Problems for Preliminary Exam
Applied Mathematics
January 2023

Partial Differential Equations

1. Solve
$$xu_x + yu_y + zu_z = 1,$$
with the condition
$$u = 0 \quad \text{on} \quad x + y + z = 1.$$
For which x, y, z the solution is defined?

2. Does there exist positive harmonic u in the unit ball in $\mathbb{R}^3$ such that
$$u(0,0,0) = 1, \quad u(0,0,1/2) = 10?$$

3. Formulate and prove the weak maximum principle for the heat equation
$$u_t = u_{xx}, \quad x \in (A,B), \quad t \in (0,T).$$

4. Solve the Laplace equation in the square $\Omega = \{x, y \mid 0 < x, y < 1\}$ with the boundary conditions
$$\begin{aligned} &u(x,0) = \sin \pi x, \quad 0 < x < 1,\\ &u(x,1) = 0, \quad 0 < x < 1,\\ &u(0,y) = \sin \pi y, \quad 0 < y < 1,\\ &u(1,y) = 0, \quad 0 < y < 1. \end{aligned}$$

5. Solve
$$u_{tt} = u_{xx} + u_{yy},$$
with the initial conditions
$$u(0,x,y) = x^2 + y^2, \quad u_t(0,x,y) = 0.$$
(Please justify each step.)

6. Let $M(t) = \int_\Omega u(x,t)\,dx$, where u satisfies the heat equation
$$u_t - \Delta u = 0 \quad \text{in } \Omega \times (0, \infty)$$

with the Neumann boundary conditions

$$\partial_{\hat{\mathbf{n}}} u = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

Show that M is constant in time.

7. Let u solve $u_{x_1 x_1} = u_{x_2 x_2}$ in $\mathbb{R}^2$ and $u(x) = 0$ for all $x \in B_1(0)$ (ball with center at the origin and radius 1). Find the largest set in $\mathbb{R}^2$ on which u is necessarily zero.