Problems for Preliminary Exam
Applied Mathematics
August 2019

Part I
All problems have 10 points.

1. (a) Pendulum equation without friction can be written as

$$\dot{x} = y, \quad \dot{y} = -a \sin x, \quad a > 0.$$

Suppose $E(x, y) = a(1 - \cos x) + \frac{1}{2}y^2$. Find a domain for which $E(x, y)$ is positive definite and then use E as a Lyapunov function to analyze stability of solution near the origin.

(b) Consider Pendulum equation with friction

$$\dot{x} = y, \quad \dot{y} = -a \sin x - by, \quad a, b > 0.$$

Can you take the same Lyapunov function as in part (a)? If not, suggest an appropriate Lyapunov function and analyze the stability of solution near the origin (do the same if you think Lyapunov function from part (a) is appropriate in this case).

2. Consider a first-order equation in $\mathbb{R}$ with $f(t, x)$ defined on $\mathbb{R} \times \mathbb{R}$. Suppose $xf(t, x) < 0$ for $|x| > R$, sor some $R > 0$. Show that all solutions exist for all $t > 0$.

3. Solve the equation

$$\ddot{x} + \omega_0^2 x = \cos(\omega t), \quad \omega_0, \omega > 0.$$

Discuss the behavior of solution as $t \to \infty$. What happens when $\omega = \omega_0$?

4. Consider the system

$$y' = Ay + g(t), \tag{1}$$

where

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad g(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Verify that

$$\Phi(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix},$$

is a fundamental matrix for $y' = Ay$. Find the solution ϕ of the non-homogeneous system (1), for which $\phi(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

5. Show that if a real homogeneous system of two first order equations has a fundamental matrix $\begin{pmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{pmatrix}$, then $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, is also a fundamental matrix. Can you find another real fundamental matrix?

Part II
All problems have 10 points.

1. Solve the equation

$$u_t + u_x + yu_y = \sin t, \quad x, t \geq 0, \quad y \in \mathbb{R}$$

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with the conditions

$$u(0, x, y) = x + y, \quad x \geq 0,$$

and

$$u(t, 0, y) = t^2 + y, \quad t \geq 0.$$

2. Give a definition of a well-posed problem. Is the problem

$$u_t = \alpha^2 u_{xx}, \quad x \in (0, l), \quad t > 0,$$

with the boundary conditions $u(t, 0) = u(t, l) = 0$, $t > 0$ and initial condition $u(0, x) = g(x)$, $x \in (0, l)$ with sufficiently smooth g well-posed? Provide arguments.

3. Solve

$$u_{tt} - c^2 u_{xx} = \cos x, \quad u(0, x) = \sin x, \quad u_t(0, x) = 1 + x.$$

4. Show that the solution to the Robin problem for the Laplace equation

$$\Delta u = 0, \quad x \in \Omega \subseteq \mathbb{R}^d,$$

with the boundary condition

$$\partial_n u + \alpha u = \beta, \quad x \in \partial\Omega$$

is unique when $\alpha > 0$.

5. Find eigenvalues and eigenfunctions for the problem

$$-X'' = \lambda X, \quad x \in (0, l), \quad X(0) = X'(l) = 0.$$

Using these eigenvalues and eigenfunctions solve the wave equation

$$u_{tt} = c^2 u_{xx}, \quad u(t, 0) = u_x(t, l) = 0$$

with the initial conditions

$$u(x, 0) = \sin(3\pi x/(2l)) - 2\sin(5\pi x/(2l)), \quad u_t(x, 0) = 0.$$