Problems for Preliminary Exam
Applied Mathematics
May 2019

Instruction: Part I (ODE) is mandatory. Please choose between Part II (PDE) and Part III (Optimization). Clearly mention which part (II or III) you are answering. No credit for "mix and match".

Part I. ODE
All problems have 10 points.

1. Discuss the equation

$$\dot{x} = x^2 - \frac{t^2}{1+t^2}.$$

Show that there is a unique solution which asymptotically approaches the line $x = 1$. Show that all solutions below the solution in the last part approach the line $x = -1$.

2. Use regular perturbation theory to approximate the solution of

$$\ddot{x} + x + \epsilon x^3 = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0,$$

up to order two.

3. Consider the equation

$$\ddot{x} - b\dot{x} - a^2 x + x^3 = 0,$$

where $x = x(t)$ and $a > 0$. Transform this to a system of first order equations. For that system, what are the conditions to obtain a saddle point at the origin and stable nodes for the other equilibrium points?

4. For a linear harmonic oscillator

$$\ddot{x} + kx = 0, \quad k > 0,$$

use an appropriate Lyapunov function to show that the origin is stable. What happens for the stability of the origin if a damping term is added to the equation as the following:

$$\ddot{x} + kx + \epsilon\dot{x}^3(1 + x^2) = 0, \quad k > 0, \quad \epsilon > 0.$$

5. Consider the nonlinear autonomous system

$$\dot{x} = -y + x(x^2 + y^2 - 625), \quad \dot{y} = x + y(x^2 + y^2 - 625).$$

Transform the equations to polar coordinate and describe the nature of solution near $(0, 0)$. Are there any limit cycles?

Part II. PDE
All problems have 10 points.

1. Compute the solution to

$$u_x + xu_y = y, \quad u(0, y) = \cos y.$$

Clearly state for which (x, y) the solution is defined.

2. Show that a harmonic function is invariant with respect to translations and rotations (that is, assuming that u is harmonic, show that $x \mapsto u(x + c)$ and $x \mapsto u(Mx)$ are also harmonic for constat c and orthogonal M, here $x \in \mathbb{R}^n$, $u\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, $c \in \mathbb{R}^n$, M is an $n \times n$ orthogonal matrix).

3. Consider the Neumann problem for the Poisson equation

$$\Delta u = f, \quad x \in \Omega, \quad \partial_n u = g, \quad x \in \partial\Omega.$$

Discuss the existence of solutions to this problem and in particular determine the conditions on f and g that will guarantee that solution does not exist.

4. Find a solution to this problem (here $d > 0$ is a constant)

$$\begin{aligned} u_t &= du_{xx}, \quad x \in (0, \pi),\, t > 0, \\ u(t, 0) &= u(t, \pi), \quad t > 0, \\ u(0, x) &= g(x), \quad x \in [0, \pi], \quad g(0) = g(\pi) = 0, \quad g \in \mathcal{C}^1. \end{aligned}$$

Is this problem well posed? Provide arguments.

5. Find a formal solution to

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\, t > 0,$$

with the initial conditions

$$u(0, x) = 1, \quad |x| < a, \quad u(0, x) = 0, \quad |x| \geq a, \quad u_t(0, x) = 0, \quad x \in \mathbb{R}.$$

Sketch profile of the solution at different time moments.
Consider the same problem with the initial conditions

$$u(0, x) = 0, \quad x \in \mathbb{R}, \quad u_t(0, x) = 1, \quad |x| < a, \quad u_t(0, x) = 0, \quad |x| \geq a.$$

Find its formal solution and sketch its profile at different time moments.

Part III. Optimization
All problems have 10 points.

1. Let a function $f\colon R \to R$ is such that $f(x) > 0$ for all $x \neq 0$ and $f(\lambda x) = \lambda f(x)$ for all $x \in R$ and $\lambda > 0$. Let μ_A be Minkowski function of set A. Prove that

$$f = \mu_{\{x\,:\,f(x)\leq 1\}}.$$

2. Find a saddle point of Lagrange function of the following convex optimization problem:

$$x_1^2 + 3x_1 \to \min, \qquad x_1^2 + x_2^2 - 2x_1 + 8x_2 + 16 \leq 0, \;\; x_1 - x_2 \leq 5.$$

3. Solve the problem:

$$\int_0^1 \dot{x}^2 dt \to \min, \quad \int_0^1 x dt = \int_0^1 tx dt = 0, \;\; x(1) = 1.$$

4. Solve the problem

$$\int_0^1 u^2 dt + \dot{x}^2(0) \to \min, \quad \ddot{x} - x = u, \;\; x(0) = 1.$$

5. Solve the problem:

$$T \to \min, \quad \dot{x}_1 = x_2 - 2, \;\; \dot{x}_2 = u, \;\; x_1(0) = x_2(0) = 0, \;\; x_1(T) = -1, \;\; x_2(T) = 0, \;\; |u| \leq 1.$$