Problems for Preliminary Exam
Applied Mathematics
January 2021

Part I. Ordinary Differential Equations
All problems have 10 points.

1. Prove that each solution (except for $x = y = 0$) of the autonomous system

$$\dot{x} = y + x(x^2 + y^2),$$
$$\dot{y} = -x + y(x^2 + y^2),$$

blows up in finite time. What is the blow up time for the solution that starts at the point $(1, 0)$ when $t = 0$?

2. Can the graphs of two solutions to $x'' = t + x^2$ intersect at some point (t_0, x_0) in the plane tOx? Can the graphs of two solutions to $x'' = t + x^2$ be tangent to each other at some point (t_0, x_0)?

3. Consider a linear system of ODE

$$\dot{x} = A(t)x + f(t),$$

with A, f continuous on $\mathbb{R}$. Prove that if one solution to this problem is stable (asymptotically stable) then any solution is stable (asymptotically stable).

4. Determine the stability properties of the origin for

$$\dot{x} = -x^3 + 2y^3,$$
$$\dot{y} = -2xy^2.$$

5. Find Green's function for the linear differential operator

$$L = \frac{d^2}{dx^2} + 1, \quad y(0) = y(\pi), \quad y'(0) = y'(\pi).$$

Part II. Partial Differential Equations
All problems have 10 points.

1. Consider the following classical Cauchy problem for the wave equation:

$$\begin{cases} u_{tt}(x,t) \,=\, a^2\, u_{xx}(x,t), \quad x \in \mathbb{R}, \quad 0 < t \leqslant T, \\ u(x,0) \,=\, \varphi(x), \quad u_t(x,0) \,=\, \psi(x), \quad x \in \mathbb{R}, \\ a = \text{const}, \quad T = \text{const} > 0, \quad \varphi \text{ and } \psi \text{ are sufficiently smooth.} \end{cases}$$

Show the continuous dependence of its solution on the initial data, i. e., verify that, if u_1 and u_2 are solutions with $(\varphi, \psi) = (\varphi_1, \psi_1)$ and $(\varphi, \psi) = (\varphi_2, \psi_2)$, respectively, then, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, T) > 0$ such that

$$\sup_{x\in\mathbb{R}} |\varphi_1(x) - \varphi_2(x)| \,<\, \delta, \quad \sup_{x\in\mathbb{R}} |\psi_1(x) - \psi_2(x)| \,<\, \delta$$

would imply

$$\sup_{x\in\mathbb{R},\ t\in[0,T]} |u_1(x,t) - u_2(x,t)| \,<\, \varepsilon.$$

2. Consider the quasilinear first-order equation

$$u(x,y)\, u_x(x,y) \,+\, u_y(x,y) \,=\, 1, \quad (x,y) \in \mathbb{R}^2,$$

with the boundary condition

$$u\left(\frac{y^2}{4}\,,\, y\right) \,=\, \frac{y}{2}\,, \quad y \in \mathbb{R}.$$

Solve this Cauchy problem and describe the region in the (x, y)–plane where there exists a classical solution. If it is not possible to obtain a unique classical solution, explain why. Also, check whether each constructed solution is differentiable on the initial data or not.

3. By using the Fourier transform, solve the following Cauchy problem for the Laplace equation:

$$\begin{cases} u_{xx}(x,y) \,+\, u_{yy}(x,y) \,=\, 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u(x,0) \,=\, \begin{cases} 1 & \text{if } x \in [-1,1], \\ 0 & \text{otherwise,} \end{cases} \\ u(x,y) \to 0 \text{ and } u_x(x,y) \to 0 \text{ as } y \to +\infty \\ \qquad\qquad \text{uniformly with respect to } x \in \mathbb{R}. \end{cases}$$

4. Let $Q = (0,\pi) \times (0,T)$, and denote the closure of this domain by $\overline{Q}$. Assume that

$u \in \mathrm{C}^2(Q) \cap \mathrm{C}^0\left(\overline{Q}\right)$ (i. e., u is twice continously differentiable in Q and continuous in $\overline{Q}$) solves the following boundary value problem for the heat equation:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + F(x,t), & (x,t) \in Q, \\ u(0,t) = g(t), \quad u(\pi,t) = 0, & t \geqslant 0, \\ u(x,0) = f(x), & x \in [0,\pi]. \end{cases}$$

a) Let

$$M = \max_{(x,t) \in \overline{Q}} \{0,\, g(t),\, f(x)\}, \qquad N = \max_{(x,t) \in \overline{Q}} \{0,\, F(x,t)\}.$$

Verify that $u(x,t) \leqslant M + Nt$ in $\overline{Q}$.

Hint. Note that the maximum principle for the heat equation allows an extension to the partial differential inequality case $v_t - v_{xx} \leqslant 0$. Take an appropriate v and use this extended principle.

b) Let $g \equiv 0$ and $F \equiv 0$. It is well-known that, in case of continuous f and f' with $f(0) = f(\pi) = 0$, the above problem admits a classical solution in class $\mathrm{C}^2(Q) \cap \mathrm{C}^0\left(\overline{Q}\right)$. Establish the existence and uniqueness of a classical solution for the case when f with $f(0) = f(\pi) = 0$ is only assumed to be continuous (i. e., when its continuous differentiability is not imposed).

Hint. The formal solution u (that can be constructed via separation of variables) is obviously smooth for $t > 0$. It hence remains to verify that u can be continuously extended to $t = 0$. Consider a functional sequence $\{f_n\}_{n=1}^{\infty} \subset \mathrm{C}^1([0,\pi])$ that uniformly converges to f and satisfies $f_n(0) = f_n(\pi) = 0$ for all n. Show that the sequence of the related classical solutions $\{u_n\}_{n=1}^{\infty}$ is fundamental and use the completeness of the space $\mathrm{C}^0\left(\overline{Q}\right)$. The related limit will give the sought-after extension. Use the Arzelá–Ascoli (bounded convergence) theorem for continuously differentiable functions to establish this. Provide all details of your reasoning.

5. Solve the following boundary value problem for the heat equation:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & 0 < x < 3, \quad t > 0, \\ u(0,t) = 0, \quad u(3,t) = 3, & t \geqslant 0, \\ u(x,0) = 4x - x^2, & 0 \leqslant x \leqslant 3. \end{cases}$$