Problems for Preliminary Exam
Applied Mathematics
May 2017

Part I
All problems have 10 points.

1. Prove that all the solutions to

$$\dot{x} = \frac{1}{1 + t^2 + x^2}$$

are bounded for all $t \in \mathbf{R}$.

2. Find the derivative of the solution to $\dot{x} = x^2 + x \sin t$ with respect to the initial condition $x(0) = a$ at $a = 0$.

3. Prove

Lemma 1 (Liouville's theorem)**.** *A Hamiltonian system conserves the phase volume.*

Recall that Hamiltonian system has the form

$$\dot{x}_i = \frac{\partial H}{\partial p_i}(x, p)\,, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}(x, p)\,, \quad i = 1, \ldots, n, \quad H \in \mathcal{C}^{(1)}(\mathbf{R}^{2n}; \mathbf{R}).$$

4. Determine the stability properties of $(x, \dot{x}) = (0, 0)$ for the equation

$$\ddot{x} + x^n = 0, \quad n \in \mathbf{N}.$$

5. Find Green's function for

$$x^3 y'' + 3x^2 y' + xy = f(x), \quad y(1) = 0, \quad y(2) + 2y'(2) = 0.$$

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1. Denote by $B_n(x,R)$, $S_n(x,R)$ an open ball and a sphere in R^n centred at x with radius R. Assume $u \in C^2(B_2(0,1)) \cap C(\overline{B_2(0,1)})$, $\Delta u = 0$ in $B_2(0,1)$, $u(x) = x_2^2$ if $x \in S_2(0,1)$, $x_2 \geq 0$, and $u(x) = x_2$ if $x \in S_2(0,1)$, $x_2 < 0$. Find

$$I = \int_{B_2(0,\frac{1}{2})} u(x)dx.$$

2. Solve the following Cauchy problem in $R^3 \times R_+$:

$$u_t = \Delta u - 3u, \quad u(x,0) = e^{-(x_1+x_2+x_3)}.$$

3. Let u be a bounded solution of the Cauchy problem

$$u_t = u_{xx}, \quad u(x,0) = \varphi(x)$$

in $R \times R_+$, where function φ is continuous and bounded in R. Assume the following limit exists and is finite

$$A = \lim_{R\to\infty} \frac{1}{R} \int_{-R}^{R} \varphi(x)dx.$$

Find $\lim_{t\to\infty} u(0,t)$.

4. Assume u is a solution of the following Cauchy problem in $R^3 \times R_+$:

$$u_{tt} = \Delta u, \quad u(x,0) = 0, \quad u_t(x,t)|_{t=0} = (1 + 4\|x\|^2)^{-1/2}.$$

Find $\lim_{t\to\infty} u(x,t)$.

5. Solve the following Cauchy problem in $R \times R_+$:

$$\frac{\partial u}{\partial x_1} - (u^3 + x_2)\frac{\partial u}{\partial x_2} + u = 0, \quad u(0,x_2) = (x_2)^{1/3}.$$