Problems for Preliminary Exam
Applied Mathematics
August 2018

Part I
All problems have 10 points.

1. Derive the Taylor series for $x(t) = \sin t$ by applying the Picard method to the first order system corresponding to

$$\ddot{x} = x, \quad x(0) = 0, \quad \dot{x}(0) = 1.$$

2. Let A be $k \times k$ real matrix, where k is odd. Show that there exists a nonperiodic solution to $\dot{x} = Ax$.

3. Study the stability properties of the trivial solution in the following problem:

$$\dot{x_1} = x_2 - x_1 + x_1 x_2,$$
$$\dot{x_2} = x_1 - x_2 - x_1^2 - x_2^3.$$

4. Is the solution to

$$\dot{x} = 4x - t^2 x, \quad x(0) = 0,$$

Lyapunov stable, asymptotically stable, or neither?

5. Prove that all the solutions to $\dot{x} = \frac{1}{1+t^2+x^2}$, are bounded for all real t.

Part II
All problems have 10 points.

1. Find the solution to

$$x^2 u_x + xy u_y = u^2,$$

which passes through the curve $u = 1, x = y^2$.

2. Solve the initial value problem

$$u_{tt} - c^2 u_{xx} = \cos x, \quad u(x,0) = \sin x, \quad u_t(x,0) = 1 + x.$$

3. Show that the operator $A = \frac{d^4}{dx^4}$ defined on the domain

$$D = \{f \in C^{(4)}[0,l] \mid f(0) = f(l) = f''(0) = f''(l) = 0\}$$

has real and nonnegative eigenvalues.

4. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Show that there exists at most one solution to

$$\Delta u = 0, \quad x \in \Omega, \quad u = f \text{ on } \partial\Omega.$$

What can you say about the same problem with the boundary condition $\partial_\nu u = f$ on $\partial\Omega$, where ν is the outward normal to the boundary of Ω?

5. Find the solution to

$$u_t = \Delta u - cu \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

with the initial condition

$$u(x, 0) = g(x) \quad \text{on } \mathbb{R}^n.$$