Problems for Preliminary Exam
Applied Mathematics
May 2016

Part I
All problems have 10 points.

1. Prove that all the solutions to

$$\dot{x} = \frac{1}{1+t^2+x^2}$$

are bounded for all $t \in \mathbf{R}$.

2. Can the graphs of two solutions of the given ODE cross on the plane (t, x)? Be tangent to each other?

$$(a) \quad \dot{x} = t + x^2, \quad (b) \quad \ddot{x} = t + x^2.$$

3. Explain clearly if an asymptotically stable equilibrium become unstable in Lyapunov's sense under linearization?

4. Determine the stability properties of the origin for the system

$$\dot{x} = -xy^4,$$
$$\dot{y} = yx^4.$$

5. For the boundary problem

$$y'' + y = f(x), \quad y(0) = y(\pi), \quad y'(0) = y'(\pi)$$

find Green's function.

Part II
All problems have 10 points.

1. Let $u(x) \geq 0$ be continuous in closed bounded domain $\bar{D} \subset \mathbb{R}^n$ and Δu is continuous in $\bar{D}$. Suppose that

$$\Delta u = u^2, \quad u\big|_{\partial D} = 0.$$

Prove that $u \equiv 0$, in D. What can you say about $u(x)$ when the condition $u(x) \geq 0$ in D is dropped?

2. Assume that U is a connected, open, bounded set. Show that constant functions are the only smooth solution of the Neumann boundary-value problem:

$$\begin{cases} -\Delta u = 0, & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial U. \end{cases}$$

3. Assume

$$\hat{u}_k \rightharpoonup \hat{u} \quad \text{weakly in} \quad L^2(0,T;H_0^1(U)),$$

and

$$\hat{u}_k' \rightharpoonup \hat{v} \quad \text{weakly in} \quad L^2(0,T;H^{-1}(U)),$$

where $U \subset \mathbb{R}^n$ is an open, bounded set. Prove that $\hat{v} = \hat{u}'$.

4. Suppose that $u(x,t)$ solves

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g, \quad u_t = h, & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

where g and h are smooth and have compact support. Show that there exists a constant λ such that

$$|u(x,t)| \leq \frac{\lambda}{t},$$

for $x \in \mathbb{R}^3$ and $t > 0$.

5. Give an example of a continuous function on $[0,1]$ which has classical derivative defined almost everywhere, but which is not weakly differentiable.