Problems for Preliminary Exam
Applied Mathematics
August 2017

Part I
All problems have 10 points.

1. How many derivatives does the solution to the following equation have in a neighborhood of the origin:

$$y' = x + y^{\frac{7}{3}}?$$

2. For which a each solution can be extended to the interval $-\infty < x < \infty$ for the equation

$$y' = (y^2 + e^x)^a?$$

3. Solve

$$y'' + 2y' + y = \cos ix.$$

Here i is the imaginary unit, $i^2 = -1$.

4. For which values of the real parameter a the equilibrium $x_1 = x_2 = x_3 = 0$ of the system

$$\dot{x}_1 = ax_1 - x_2,$$
$$\dot{x}_2 = ax_2 - x_3,$$
$$\dot{x}_3 = ax_3 - x_1,$$

is stable?

5. Find Green's function for

$$y'' \cos^2 x - y' \sin 2x = f(x), \quad y(0) = y'(0), \quad y(\pi/4) + y'(\pi/4) = 0.$$

Please turn over...

Part II
All problems have 10 points.

1. Does there exist a function u harmonic in the ball $B = \{x \in R^3 : \|x\| \leq 1\}$ such that $u(x) \geq 0$ for all $x \in B$, $u(0,0,0) = 1$ and $u(0,0,\frac{1}{2}) = 10$?

2. Let u be a solution of the heat equation

$$u_t = \Delta u$$

in $R^3 \times (0,1)$ such that $u(x,t) \geq 0$ for all $(x,t) \in R^3 \times [0,1]$ and $u(x,t) = 0$ in the cube $[0,1] \times [0,1] \times [0,1] \times [0,1]$. Is it true that $u(x,t)$ has to be zero function in $R^3 \times [0,1]$?

3. Let u be a solution of the following Cauchy problem in $R \times R_+$:

$$u_t = u_{xx}, \quad u(x,0) = \frac{x^4 + \cos x}{2 + 3x^4}.$$

Find $\lim_{t\to\infty} u(x,t)$.

4. Assume u is a solution of the following Cauchy problem in $R^3 \times R_+$:

$$u_{tt} = \Delta u, \quad u(x,0) = 0, \quad u_t(x,t)|_{t=0} = \psi(x),$$

where $\psi(x) = 0$ if $\|x\| \in [0.9, 1.0]$, and $\psi(x) > 0$ for all other x. Find all points $(x,t) \in R^3 \times R_+$ such that $u(x,t) = 0$.

5. Solve the following Cauchy problem in $R_+ \times R$:

$$\frac{\partial u}{\partial x_1} + (u + x_2)\frac{\partial u}{\partial x_2} + u = 0, \quad u(0,x_2) = x_2.$$