Problems for Preliminary Exam
Applied Mathematics
January 2022

Ordinary Differential Equations

1. Consider the following IVP

$$\dot{x} = \max\{1, x\}\,, \quad x(t) \in \mathbb{R}, \quad x(0) = 0.$$

Show that the solution to this problem exists and unique. After it find this solution.

2. Let

$$\dot{x} = a(t)x, \quad x(t) \in \mathbb{R},$$

where a is a continuous periodic function with minimal period $T > 0$. Define

$$\lambda = e^{\int_0^T a(s)ds}.$$

Formulate the definitions for the trivial solution to be Lyapunov stable, asymptotically stable, and unstable.

After this prove that if $\lambda = 1$ then the trivial solution is Lyapunov stable, if $\lambda < 1$ then it is asymptotically stable, if $\lambda > 1$ then it is unstable.

3. Show that $(0, 0)$ is an asymptotically stable equilibrium of the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - (1 - x^2)y,\end{aligned}$$

and that the basin of attraction of $(0, 0)$ contains the unit disk.

4. List a basis of the space of solutions to

$$x^{(\text{iv})} - 2x'' + x = 0.$$

5. For which real parameters a, b the trivial solution to

$$\begin{aligned}\dot{x} &= x + ay + y^2,\\ \dot{y} &= bx - 3y - x^2,\end{aligned}$$

is asymptotically stable? (Do not analyze the non-hyperbolic case.)

6. For which real a the BVP

$$y'' + ay = f(x), \quad y(0) = y(1) = 0$$

admits existence of Green's function?

7. Consider the IVP

$$\dot{x} = t^3 - x^3, \quad x(t) \in \mathbb{R}, \quad x(t_0) = x_0.$$

Prove that for any $(t_0, x_0) \in \mathbb{R}^2$ the solution to this problem can be always extended to $[t_0, +\infty)$ and there exist solutions that cannot be extended to $(-\infty, t_0]$.

8. Let

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}.$$

Is it possible to find a two by two real matrix B such that

$$A = e^B?$$