Math747 Preliminary Exam

August 2023

Note: "Manifold" refers to a manifold **without** boundary.

Problem 1. Let $\omega = (3y^2 - 4xz)dx - 3yzdy + (2x - y - z)dz$ be a differential 1-form in $\mathbb{R}^3$.

a) Is ω exact? Is it closed?

b) Find $\omega \wedge d\omega$.

c) Is there a smooth function $f(x, y, z)$ such that $df \wedge \omega = d\omega$?

Problem 2. a) Find an immersion of $\mathbb{R}P^2$ in $\mathbb{R}^3$.

b) Find an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.

Problem 3. Let M, N be smooth manifolds.

a) For a smooth map $f : M \to N$ define the notions of rank and critical value.

b) State Sard's Theorem.

c) Prove Sard's Theorem in the one-dimensional case (i.e. when $dimM = dimN = 1$).

Problem 4. Let M be a smooth manifold and $f : M \to \mathbb{R}^n$ be a smooth map for some $n \geq 1$.

a) Prove that if f is a submersion, then M is not compact.

b) Prove that if f is an immersion and M is compact, then $dimM < n$.

Problem 5. Let $c \in \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}^3$ be given as

$$F(x, y) = (cxy, x^2 + cy^2, x + y).$$

a) For what real values of c is F an immersion?

b) For what real values of c is F an embedding?