NDSU MATHEMATICS DEPARTMENT
Geometry and Topology Qualifying Exam. August 18th, 2017.

Unless otherwise stated "manifold" refers to a manifold **without** boundary.

Problem 1. A topological space X is called *locally Euclidean* if for some fixed $n \geq 1$ and for all $p \in X$ there exists an open set U containing p such that U is homeomorphic to $\mathbb{R}^n$. Is it true that a locally Euclidean second countable space is Hausdorff?

Problem 2. Let X be a Hausdorff topological space and U an open subset of X. Show that if $\{K_n : n \in \mathbb{N}\}$ is a nested sequence of compact subsets of X with $\bigcap_{n=1}^{\infty} K_n \subseteq U$, then there exists $N \in \mathbb{N}$ such that $K_N \subseteq U$.

NOTE: "nested" means that $K_{n+1} \subseteq K_n$ for all $n \geq 1$.

Problem 3. On the manifold $\mathbb{R}^2 \backslash \{0\}$, please:

a) Find a closed one-form that is not exact.

b) Show that any compactly supported closed one-form is exact. [The support of a differential form is the point set on which it does not vanish.]

Problem 4. a) Define *transversal intersection* of two submanifolds.

b) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are C^1 functions and that the derivatives $df_x \neq dg_x$ whenever $f(x) = g(x)$. Prove that the graphs of f and g are submanifolds of $\mathbb{R}^{n+1}$ and that they intersect transversally. Note: the graph of f is $\{(x, f(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$.

Problem 5. *Find an explicit map* to show that the Klein bottle embeds into $\mathbb{R}^4$.

Problem 6. For what values of c is the set

$$\mathcal{V}_c = \{(x, y, z) : x^3 + y^3 + z^3 = c, xy = z\}$$

a smooth submanifold of $\mathbb{R}^3$?

Problem 7. Calculate $H_*(X; \mathbb{Z})$ where

$$X = S^2 \cup \{(0, 0, t) \in \mathbb{R}^3 | -1 \leq t \leq 1\} \cup (D^2 \times \{0\}),$$

and $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}, D^2 = \{(x, y) : x^2 + y^2 \leq 1\}$. In words: X is the union of a 2-sphere with an equatorial disk and with a line segment joining the North and South poles.

Problem 8. Consider the capital letters of the alphabet, as below, in the sans serif style with no adornments:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Consider each letter as a topological space endowed with the subspace topology inherited from $(\mathbb{R}^2, \textit{Euclidean})$.

a) Prove that X is not homeomorphic to Y.

b) Consider the equivalence relation "is homeomorphic to" on the set of these letters. What are its equivalence classes? Justify your choices.

c) Consider the equivalence relation "is homotopically equivalent to" on the set of these letters. What are its equivalence classes? Justify your choices.

Problem 9. Determine, with proof, the number of connected 2:1 coverings of the wedge sum $S^1 \vee S^1 \vee S^1$.

Problem 10. Let M and N be smooth manifolds, and suppose $F : M \to N$ is an injective smooth map of constant rank. Prove that if M is compact, then F is an embedding.