NDSU Mathematics Department

GEOMETRY and TOPOLOGY QUALIFYING EXAMINATION,
May 15th, 2018

Unless otherwise stated, "manifold" refers to a manifold without boundary.

Section A - Differential Topology:

1. a. Give an example of a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ which is not a diffeomorphism.
   b. Prove or disprove: Let $F : M \to N$ be a smooth map between smooth manifolds M and N and let $c \in N$ be a critical value of F. Then $F^{-1}(c)$ is not an embedded submanifold of M.

2. Suppose (x, y) are Cartesian coordinates on $\mathbb{R}^2$ and (u, v) are new coordinates defined by
$$u = 5x + y - 3 \text{ and } v = -x + y + 5.$$
   Express the vector field $x\frac{\partial}{\partial x}$ as a vector field $P(u, v)\frac{\partial}{\partial u}+Q(u, v)\frac{\partial}{\partial v}$.

3. Let $M = \mathbb{R}^3\backslash S^2$. For which values of k does there exist a smooth k-form ω on M such that $d\omega = 0$, but $\omega \neq d\tau$ for any $(k-1)$-form.

4. Let M^m be a compact smooth manifold and $E \to M$ a vector bundle of rank k. Prove the following:
   a. If $k > m$, then E admits a nowhere vanishing section.
   b. If $k \leq m$, then E admits a section σ such that the set of points where σ vanishes is a smooth compact codimension k submanifold of M.

5. Let M be a smooth manifold of dimension $m < n$. Let f be a smooth map $f : M \to S^n$. Prove that f is smoothly homotopic to a constant map.

Section B - Algebraic Topology:

1. a. Let X be a Hausdorff topological space. Show that there exists a compact topological space Y such that X is a subspace of Y and $Y\backslash X$ consists of a single point.
   b. For distinct positive integers m and n, prove that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.

2. A topological space is said to have *the fixed-point property* if all continuous maps $f : X \to X$ fix a point.
   a. Show that $\mathbb{S}^n$ does not have the fixed-point property for all $n \geq 0$.
   b. Prove that $\mathbb{R}P^2$ has the fixed-point property.
   c. Which compact orientable 2-manifolds have the fixed-point property?

3. Let m, n be positive integers. Find homology groups of
   a. $\mathbb{S}^m \times \mathbb{S}^n$
   b. $\mathbb{S}^m \vee \mathbb{S}^n$

4. Show that for all $k \geq 3$ the free group $\mathbb{F}_k$ is isomorphic to a subgroup of $\mathbb{F}_2$.

   HINT: Use covering spaces.

5. a. Find $\chi(\mathbb{T}^n)$.
   b. Let X, Y be finite CW-complexes. If $\chi(X) = \chi(Y) = 0$ is it true that $\chi(X \times Y) = 0$?