NDSU Mathematics Department

GEOMETRY and TOPOLOGY QUALIFYING EXAMINATION

August 16, 2018

Please write all answers using a blue pen. Only write on one side of the paper. Clearly cross-out any work which you do NOT want us to grade.

Unless otherwise stated, "manifold" refers to a manifold without boundary.

Part A:

(1) Find a topological space which is path connected but not locally path connected.

(2) Let M be a connected manifold, and p, q be points of M. Prove that there exists a homeomorphism $f : M \to M$ such that $f(p) = q$.

(3) Let X and Y be finite CW complexes such that X covers Y. Prove that $\chi(X)$ is an integer multiple of $\chi(Y)$.

(4) Let n be a positive integer.
  (a) Determine the homology groups of $\mathbb{C}P^n$.
  (b) Dtermine $\pi_1(\mathbb{C}P^n)$.
  (c) Use the Mayer-Vietoris sequence and induction to compute the de Rham cohomology groups of $\mathbb{C}P^n$.

(5) A topological space X is obtained by identifying all four vertices of a tetrahedron. Determine the homology groups of X.

Part B:

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^3 + xy + y^3$.
  (a) Show that $f^{-1}(1)$ is a smooth submanifold of $\mathbb{R}^2$.
  (b) Show that $f^{-1}(0)$ is not a smooth submanifold.(Hint: If $(x(t), y(t))$ is a curve in $f^{-1}(0)$ with $x(0) = y(0) = 0$, then what is the condition on $x'(0)$ and $y'(0)$?)

(2) Let M be a compact, connected and orientable smooth manifold of dimension 6. Let α and β be two-forms on M. Show that there is a point of M where $d\alpha \wedge d\beta = 0$.

(3) Let $M(n)$ denote the space of $n \times n$ matrices with real entries. Prove that the orthogonal group

$$O(n) = \{A \in M(n) \mid AA^t = Id_n\}$$

is a manifold of dimension $n(n-1)/2$.

(4) Define a 1-form on $\mathbb{R}^2 \backslash \{0\}$ by

$$\omega = -\left(\frac{y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

Calculate $\int_C \omega$ for any simple closed curve containing the origin.

(5) For a plane $P \subset \mathbb{R}^3$, let $\pi_P : \mathbb{R}^3 \to P$ denote the orthogonal projection onto P. Suppose that $g : S^1 \to \mathbb{R}^3$ is a smooth embedding. Prove that there exists a plane P for which $\pi_P \circ g$ is an immersion.