NDSU Mathematics Department
GEOMETRY and TOPOLOGY QUALIFYING EXAMINATION
May 15, 2019

Please write all answers using a blue pen. Only write on one side of the paper. Clearly cross-out any work which you do NOT want us to grade.

Unless otherwise stated, "manifold" refers to a manifold without boundary.

Part A

A1 (1) Give an example of a topological space which is path connected but not locally path connected.
(2) Give an example of a topological space which is locally path connected but not locally simply connected.

A2 A topological space X is obtained as a quotient of the union of a hexagon $aba^{-1}cbc$ and an octagon $c^2aba^{-1}b^3$ by gluing all the corresponding sides. Compute the homology groups of X.

A3 Find the fundamental group of
(1) $\mathrm{SL}(2,\mathbb{R})$
(2) $\mathrm{SO}(3,\mathbb{R})$
(Hint: Show that $\mathrm{SO}(3,\mathbb{R})$ is homeomorphic to $\mathbb{R}P^3$.)

A4 (1) For every integer n, find a map $f : S^2 \to S^2$ of degree n.
(2) For every integer n, find a map $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ of degree n^2.

A5 Let $g, h \geq 0$. Describe all pairs (g, h) such that Σ_g covers Σ_h.

Part B

B1 Consider the following statement: If ω is a smooth k-form on a smooth manifold M^n, then $\omega \wedge \omega = 0$. Discuss this statement.

B2 Show that the subset of $\mathbb{R}^3$ defined by the equation

$$(1 - z^2)(x^2 + y^2) = 1$$

is a smooth manifold.

B3 Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F(x, y) = (x + y^2, x^2).$$

Denoting by u, v the Cartesian coordinates of the target space, determine

$$F^*(v\ du + dv).$$

B4 Let

$$M = \{([x_0 : x_1 : x_2], t) \in \mathbb{R}P^2 \times \mathbb{R} \mid x_0 + x_1 t + x_2 t^2 = 0\}.$$

(1) Show that M is an embedded submanifold of $\mathbb{R}P^2 \times \mathbb{R}$.
(2) Let $\pi : M \to \mathbb{R}P^2$ be projection onto the first factor. Find the regular values of π.

B5 Let M be the smooth 3-manifold obtained by identifying $\{0\} \times S^2$ and $\{1\} \times S^2$ in $[0, 1] \times S^2$ via the map $(0, x) \mapsto (1, -x)$ for any $x \in S^2$. Compute the de Rham cohomology groups of M.