NDSU
Geometry/Topology Preliminary Examination

18 August 2016

Instructions

Please attempt all questions. Show your work.
Unless stated otherwise, all topologies and smooth structures are the standard ones.

Grading

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

Questions

1. (10 points) Let $f : M \to N$ be a smooth map between smooth manifolds M and N.
   (a) Define what it means for f to be: (i) an immersion; (ii) a submersion; and (iii) an embedding.
   (b) Show that if M is compact and f is an injective immersion, then f is an embedding.

2. (10 points) Calculate the de Rham cohomology ring of $(S^1 \times S^3)\#\mathbb{C}P^2$.

3. (10 points) Let M be a smooth manifold of dimension $2n$. We say that a 2-form $\omega \in \Omega^2(M)$ on M is *symplectic* if $d\omega = 0$ and $\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$ is a nowhere vanishing $2n$-form on M.
   (a) Let $\mathbb{R}^{2n} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \mid x_i, y_i \in \mathbb{R} \text{ for all } i\}$. Show explicitly that

   $$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

   is an exact symplectic form on $M = \mathbb{R}^{2n}$.
   (b) Show that if M is compact with no boundary then no symplectic form ω on M is exact.

4. (10 points) Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus. Let $\pi : \mathbb{R}^2 \to M$ be the quotient map.
   (a) Let $l = \mathbb{R}\cdot(7,3)$ be a line in $\mathbb{R}^2$ and let $S = \pi(l) \subset M$. Show that S is a compact embedded submanifold of M.
   (b) Find a closed differential 1-form α on M such that
$$\int_S \alpha = 1.$$
   (c) Give an example of a line l in $\mathbb{R}^2$ such that $\pi(l)$ is NOT a compact embedded submanifold of M. Briefly justify your answer, no explicit proof is needed.

5. (10 points) Let M be a compact smooth n-manifold and $f : M \to \mathbb{R}^{n+1}$ smooth with $0 \notin f(M)$. Show that there exists a line $l \subset \mathbb{R}^{n+1}$ through the origin that meets $f(M)$ in finitely many points.

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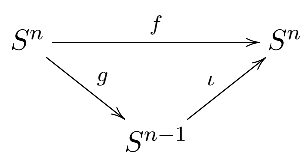
6. (10 points) Let $A, B \subset \mathbb{R}$ be closed, non-empty, disjoint sets. Prove that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f|A \equiv 1$ and $f|B \equiv -1$.

7. (10 points) Let $C \subset [0,1]$ be the Cantor middle-third subset. Let $A \subset C$ be a connected component. Prove that A is a singleton.

8. (10 points) Let S^n be the n-dimensional sphere. Prove that if a continuous map $f : S^n \to S^n$ factors into continuous maps



then f is null-homotopic.

9. (10 points) Let G be the group generated by α, β, γ subject to the relations
$$[\alpha, \beta] = \gamma, \qquad [\beta, \gamma] = [\gamma, \alpha] = 1.$$
Construct a compact, connected Hausdorff space X such that $\pi_1(X) = G$. [Be sure to prove that $\pi_1(X)$ is isomorphic to G].

10. (10 points) Let $X = \{(x, y) \mid x, y \in \mathbb{R}^3,\ |x| = 1, x \perp y\}$. Prove that X is homotopy equivalent to S^2.